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The Schottky–Klein prime function: a theoretical and computational tool for applications

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This article surveys the important role, in a variety of applied mathematical contexts, played by the so-called Schottky–Klein (S–K) prime function. While it is a classical special function, introduced by 19th century investigators, its theoretical significance for applications has only been realized in the last decade or so, especially with respect to solving problems defined in multiply connected, or 'holey', domains. It is shown here that, in terms of it, many well-known results pertaining only to the simply connected case (no holes) can be generalized, in a natural way, to the multiply connected case, thereby contextualizing those well-known results within a more general framework of much broader applicability. Given the wide-ranging usefulness of the S–K prime function it is important to be able to compute it efficiently. Here we introduce both a new theoretical formulation for its computation, as well as two distinct numerical methods to implement the construction. The combination of these theoretical and computational developments renders the S–K prime function a powerful new tool in applied mathematics.

Keywords: Schottky–Klein prime function; multiply connected; potential theory; boundary integral methods.

1. Introduction

Special functions have always played a pivotal role in the applied sciences and, while the heyday of their study has arguably passed, they remain a powerful and ubiquitous tool. Indeed the most cited mathematical work of all time is Abramowitz and Stegun's *Handbook of Mathematical Functions* (Abramowitz & Stegun, 1972), which in recent years has received around 2,000 citations per year. In a 10-year project, this classic monograph has been entirely revamped as the 'Digital Library of Mathematical Functions' (DLMF) which is both an online resource (http://dlmf.nist.gov) and a textbook reference called the *NIST Handbook of Mathematical Functions* (Olver *et al.*, 2010).

The abiding appeal of special functions is that they are characterized by their inherent mathematical properties which are well known and well documented. These functions then serve as ‘building blocks’ for more complicated functions which have their own intrinsic properties and, in particular, might form the solution of some problem of physical interest. The ability to express the solution of a problem in terms of recognized special functions that are readily computable remains among the desiderata across scientific disciplines not least because it promotes the ease with which the results of a calculation can be recorded, reproduced and analysed. Special function packages are part and parcel of all popular mathematical software tools such as MATLAB, Maple and Mathematica.

In recent years the first author, with various collaborators, has both established and given extensive evidence of the usefulness in applications of a little-known, but nonetheless classical, special function called the ‘Schottky–Klein prime function’ (henceforth, S-K prime function, or just ‘prime function’, for brevity; we have adopted the designation ‘Schottky-Klein prime function’ following Baker (1897); see also Burnside (1891, 1892)). So little known is this function, and so new are the recent developments that make use of it in applications, that it does not, in fact, appear in the newly updated DLMF or indeed in any extant special function packages. Even so, recognition of its usefulness is already gaining traction in the scientific community.

The principal motivation for studying the S-K prime function was the realization of its importance for physical science problems involving multiply connected domains, by which we mean finitely connected planar domains generally unpossessing of any geometrical symmetries. These occur in applications all the time: any problem involving multiple two-dimensional objects/entities interacting in some ambient field involves a multiply connected domain. The mathematical approach to such problems (Crowdy, 2008a, 2012) has been to develop a constructive framework based on the Schottky uniformization of the so-called Schottky double—a compact Riemann surface naturally associated to any given multiply connected planar domain—and then to make use of the S-K prime function that lives naturally on that surface as the basic functional building block for more complicated functions defined in those same domains.

This idea has proven to be surprisingly fruitful. A survey of many results in this vein, including applications to potential theory (and, hence, all the areas of physics to which it applies), the uniformization of multiply connected quadrature domains (which have many physical applications (Crowdy, 2005)), the solution to a variety of problems in fluid dynamics and solid mechanics and the derivation of a multiply connected version of the classical Schwarz–Christoffel conformal mapping formula (a well-used general tool for applications (Driscoll & Trefethen, 2002)) is given in the review articles of Crowdy (2008a, 2012). Most recently, the relevance of the S-K prime function for the study of random normal matrices has been pointed out in Crowdy (2014).

With so many uses of the S-K prime function in disparate applications the matter of having available fast and flexible computational routines for its accurate evaluation has become pressing. The purpose of this article is to describe a new numerical scheme for the computation of the S-K prime function, one that relies only on the numerical solution of a sequence of well-recognized boundary value problems. Indeed, the principal new mathematical result given here is to show how to combine a sequence of solutions to so-called ‘modified Schwarz problems’ to construct the prime function for multiply connected circular domains. The latter are an important canonical class of domains.

A previous paper (Crowdy & Marshall, 2007a) has already given an alternative (and effective) numerical scheme to compute the prime function—the first known scheme to avoid evaluation of the function based on various (often poorly convergent) infinite product representations given by the classical authors. The results of this article are viewed as superseding the prior approach of Crowdy & Marshall (2007a). The principal advantage of the new approach herein is that the numerical scheme is now built on the solution of standard boundary value problems whose numerical solution by standard methods (e.g. boundary integral methods) is by now well established.

The plan of the article is as follows. In Sections 2–4 we present some background explaining what the S-K prime function is, its defining properties and some of its uses. In Section 5 we outline some aspects of potential theory that will be needed in the development of Section 6 which gives details of the new numerical algorithm we propose for computation of the prime function. In Sections 7 and 8 we implement the numerical method using two distinct computational schemes: the first is a novel approach (an alternative to standard boundary integral methods) that makes use of Fourier representations of unknown data on the domain boundaries and use of so-called ‘global relations’ to determine that unknown data; the second implementation is closer to a standard boundary integral calculation but is less traditional in that it makes use of the generalized Neumann kernel in formulating the boundary integral equations. Both methods are shown to facilitate fast and accurate calculation of the prime function; cross-checking the results of the two separate methods afford a verification on the formulation and this is done in Section 9. Finally, in Section 10, we showcase just a few of the many possible physical applications of this new computational tool.

To accompany this article, and in the spirit of promoting wider use of the S-K prime function in applications, we have prepared easy-to-use software packages (based on the new formulation of this article) on the MATLAB platform (Crowdy, 2016b; Kropf, 2015; Nasser, 2016). These packages enable the user to readily compute the S–K prime function.

2. The simplest prime function

The S-K prime function is a classical mathematical object discussed, e.g. in Baker’s 19th century monograph on abelian functions (Baker, 1897), but it is not well known. Every mathematician—and likely every scientist—has, however, come across its simplest instance since it underpins a result as basic as the fundamental theorem of algebra. The latter states that any Nth degree polynomial $P_N(\zeta)$, with $N \geq 1$, can be uniquely factorized into a product of simpler functions of the form

$$P_N(\zeta) = \zeta^N + a_{N-1}\zeta^{N-1} + \cdots + a_1\zeta + a_0 = \prod_{k=1}^{N}(\zeta - \gamma_k), \tag{2.1}$$

where $\{\gamma_k | k = 1, ..., N\}$ are the roots of the polynomial. It is natural to define the simple monomial function of two variables, $\omega(\zeta, \gamma) \equiv (\zeta - \gamma)$, to be a ‘prime function’ because, in analogy with the fact that any integer can be factorized into a unique product of prime integers, any polynomial can be uniquely factorized into a product of such prime functions, viz,

$$P_N(\zeta) = \prod_{k=1}^{N} \omega(\zeta, \gamma_k). \tag{2.2}$$

By extension, any rational function $R(\zeta)$—a function whose only singularities in the extended complex plane (or Riemann sphere) are poles—can be written in the form

$$R(\zeta) = \prod_{k=1}^{N} \omega(\zeta, \alpha_k) \Big/ \prod_{k=1}^{N} \omega(\zeta, \beta_k), \tag{2.3}$$

where $\{\alpha_k | k = 1, ..., N\}$ are the zeros and $\{\beta_k | k = 1, ..., N\}$ are the poles of the function. The S-K prime function is the name given to the function which replaces the simple monomial function $(\zeta - \gamma)$ when the underlying compact Riemann surface has genus greater than zero. Any meromorphic function $R(\zeta)$ on such a surface then also has a representation very much akin to (2.3) that explicitly reflects the positions of its zeros and poles.

The prime function for general compact Riemann surfaces was first considered by Schottky (1887) and Klein (1890). It is discussed by Burnside (1891) and treated in a special chapter of the classic monograph by Baker (1897). It has close mathematical connections with the notion of a 'prime form' (Fay, 1973) on the Jacobi variety associated with a compact Riemann surface, and prime forms have over the years found much application in, e.g. algebraic geometry, mathematical physics and integrable systems theory.

The S-K prime function arising naturally within the Schottky model of algebraic curves has, by contrast, been used much less often and, in particular, its relevance to analysis in multiply connected domains has only very recently been formulated and explored, principally by the first author and his group. Hejhal (1972) considers the S-K prime function in his discussion of the classical kernel functions of planar domains, and it is on the particular application of the prime function to planar domains that the present article will focus. It is possible to associate with any multiply connected planar domain a compact symmetric Riemann surface called its 'Schottky double' (Gustafsson, 1983). The S-K prime function on such symmetric Riemann surfaces has certain special properties which will be reviewed here. Consequently, many results concerning the function theory of a planar domain can be conveniently expressed in terms of the S-K prime function on its Schottky double.

3. A more complicated prime function

The S-K prime function for the Riemann sphere is simple and familiar. The prime function for a sphere with one handle, or the torus, is more interesting and provides an instructive window on the higher genus case. One mathematical model of a torus is to consider the two neighbouring annuli $\rho < |\zeta| < 1$ and $1 < |\zeta| < 1/\rho$, where $0 < \rho < 1$ is a real parameter; these two annuli will model the two symmetric 'sides' of the Schottky double. They already meet at the circle $|\zeta| = 1$ but we also want them to be associated at the two other boundary circles $|\zeta| = \rho$ and $|\zeta| = 1/\rho$ (which, we note, are reflections of each other in the circle $|\zeta| = 1$). A holomorphic identification of these two circles is provided by the Möbius mapping $\zeta \mapsto \rho^2\zeta$. A meromorphic function $F(\zeta)$ on this torus can be defined as a function satisfying the functional relation

$$F(\rho^2\zeta) = F(\zeta) \tag{3.1}$$

and having only poles in the annulus $\rho \leq |\zeta| < \rho^{-1}$. As a function defined in the extended complex plane by the functional relation (3.1) $F(\zeta)$ will have poles in all other so-called 'equivalent annuli' obtained by repeatedly mapping the annulus $\rho \leq |\zeta| < \rho^{-1}$ under the transformation $\zeta \mapsto \rho^2\zeta$ or its inverse $\zeta \mapsto \rho^{-2}\zeta$ (these will produce a tessellation of the plane excluding singular points at $\zeta = 0$ and ∞). Meromorphic functions satisfying (3.1) have been dubbed 'loxodromic functions' (Valiron, 1966).

But how do we construct functions satisfying (3.1)? Consider $P(\zeta)$ defined by the infinite product

$$P(\zeta) \equiv (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^{2k}\zeta)(1 - \rho^{2k}\zeta^{-1}). \tag{3.2}$$

(Notice that this function depends on the parameter ρ, but our notation does not include this explicitly as an argument of the function.) Using standard methods for infinite products (Valiron, 1966) the function (3.2) can be shown to be absolutely convergent for all $\zeta \neq 0, \infty$ and $0 < \rho < 1$. It is easy to confirm, directly from this definition, that $P(\zeta)$ satisfies the functional relation

$$P(\rho^2\zeta) = -\zeta^{-1}P(\zeta). \tag{3.3}$$

The function $P(\zeta)$ does not itself satisfy (3.1), but the ratio of products

$$R(\zeta) \equiv \prod_{k=1}^{N} P(\zeta\alpha_k^{-1}) \Big/ \prod_{k=1}^{N} P(\zeta\beta_k^{-1}) \tag{3.4}$$

does satisfy (3.1) provided the parameters $\{\alpha_k, \beta_k | k = 1, ..., N\}$, which are all points inside the annulus $\rho \leq |\zeta| < \rho^{-1}$, satisfy the single condition

$$\prod_{k=1}^{N} \alpha_k = \prod_{k=1}^{N} \beta_k. \tag{3.5}$$

This result can be established by a simple exercise based on repeated use of (3.3). By inspection, $R(\zeta)$ can be seen to have only poles in the annulus $\rho \leq |\zeta| < \rho^{-1}$ and is therefore meromorphic on the torus. On comparing (3.4) with (2.3) it is natural to identify the function $P(\zeta)$ with the prime function for the torus and, up to normalization by a multiplicative constant, this is indeed the case.

An important observation is that since $P(\zeta)$ is analytic in the annulus $\rho < |\zeta| < 1$ then, in addition to the infinite product expression (3.2), it also has a convergent Laurent series there. By making use of (3.3) this can be shown to be given by the rapidly convergent series

$$P(\zeta) = A \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)} \zeta^n, \tag{3.6}$$

where

$$A = \prod_{n=1}^{\infty} (1 + \rho^{2n})^2 \Big/ \sum_{n=1}^{\infty} \rho^{n(n-1)}. \tag{3.7}$$

The Laurent series (3.6) converges everywhere in the annulus $\rho \leq |\zeta| < \rho^{-1}$. These two representations of the same function furnish the identity

$$(1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^{2k}\zeta)(1 - \rho^{2k}\zeta^{-1}) = A \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)} \zeta^n, \tag{3.8}$$

which relates an infinite product to an infinite sum. Indeed, as discussed in Crowdy (2008a), the function $P(\zeta)$ can be related to the first Jacobi theta function and the relation (3.8) is essentially the Jacobi triple product identity (Watson & Whittaker, 1927).

4. The S–K prime function for multiply connected circular domains

In Sections 2 and 3 we have described the prime function associated with two of the simplest circular domains: the unit disc and a concentric annulus. But it is possible to associate a S–K prime function to any multiply connected circular domain. We continue to denote such a function simply by $\omega(\zeta,\alpha)$ even though, now, the reader must remember that it will also depend on the parameters characterizing the circular domain on which it is defined.

Let D_ζ be a multiply connected circular domain comprising the interior of the unit ζ-circle, which we denote by C_0, with $m \geq 1$ smaller circular discs excised. The boundaries of the m smaller interior circles will be denoted by $\{C_j | j = 1, ..., m\}$. The concentric annulus considered earlier corresponds to the case $m = 1$. Figure 1 shows an example with $m = 2$. The centre and radius of the circle C_j will be denoted by δ_j and q_j, respectively. By extension, for C_0, we also define $\delta_0 = 0$ and $q_0 = 1$. The entire boundary of D_ζ is denoted by ∂D_ζ.

We define m Möbius maps $\{\phi_j | j = 1, ..., m\}$ by requiring that $\overline{\zeta} = \phi_j(\zeta)$ on circle C_j. That is, if C_j is defined by

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\overline{\zeta} - \overline{\delta_j}) = q_j^2 \tag{4.1}$$

then

$$\overline{\zeta} = \overline{\delta_j} + \frac{q_j^2}{\zeta - \delta_j} \tag{4.2}$$

so we can read off the fact that

$$\phi_j(\zeta) \equiv \overline{\delta_j} + \frac{q_j^2}{\zeta - \delta_j} \qquad j = 1, ..., m. \tag{4.3}$$

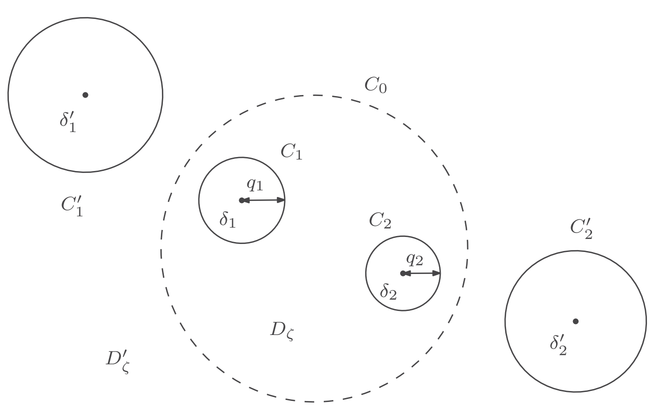

FIG. 1. Schematic of a circular domain D_ζ with $m = 2$ showing its defining parameters $\{q_j, \delta_j | j = 1, ..., m\}$; it is the interior of the unit circle C_0 with m smaller circular discs excised. The domain D_ζ' is the reflection of D_ζ in C_0; each circle C_j' is the reflection of C_j in C_0.

Armed with the set $\{\phi_j | j = 1, ..., m\}$ we now introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2 \zeta}{1 - \overline{\delta}_j \zeta} \qquad j = 1, ..., m. \tag{4.4}$$

Now let C_j' be the circle obtained by reflection of the circle C_j in the unit circle C_0, i.e. C_j' is the circle obtained by subjecting points on C_j to the transformation $\zeta \mapsto 1/\overline{\zeta}$. It is easily verified that the image of the circle C_j' under the Möbius map $\theta_j(\zeta)$ is the circle C_j; thus $\theta_j(\zeta)$ identifies circle C_j' with circle C_j. Since the m circles $\{C_j | j = 1, ..., m\}$ are non-overlapping, so are the m circles $\{C_j' | j = 1, ..., m\}$. The (classical) 'Schottky group' Θ is defined to be the infinite free group of Möbius maps generated by compositions of the m basic Möbius maps $\{\theta_j | j = 1, ..., m\}$, their inverses $\{\theta_j^{-1} | j = 1, ..., m\}$ and the identity map. The centres and radii of the reflected circles $\{C_j' | j = 1, ..., m\}$ will be denoted by δ_j' and q_j', respectively. The unbounded, open domain exterior to the circles $\{C_0, C_1', \ldots, C_m'\}$ will be denoted D_ζ', and its boundary by $\partial D_\zeta'$.

Consider the (generally unbounded) region of the plane exterior to the $2m$ circles $\{C_j, C_j' | j = 1, \ldots, m\}$. Let this region be denoted F; it is a fundamental region of the Schottky group Θ. This is because the entire plane is tessellated with copies of this fundamental region obtained by mapping F by the elements of the Schottky group. The fundamental region F can be understood as having two 'halves'—the half that is inside the unit circle but exterior to the circles C_j is the domain D_ζ, the other half, D_ζ', is the region outside the unit circle and exterior to the circles C_j'. This other half (or copy of D_ζ) is obtained by an (antiholomorphic) reflection of D_ζ in the unit circle C_0.

For readers familiar with the usual nomenclature associated with the theory of compact Riemann surfaces we point out that these two halves of F, one just a reflection through the unit circle of the other, can be viewed as a model of the front and back sides of the Schottky double (Gustafsson, 1983). The circles $\{C_j | j = 1, ..., m\}$ can be identified with a set of a-cycles of this compact Riemann surface (by identification, the circles $\{C_j' | j = 1, ..., m\}$ also correspond to a-cycles); any line joining identified points on C_j and C_j' can be identified with a b-cycle (Crowdy & Marshall, 2007a; Crowdy, 2010b). There are m such b-cycles. Any compact Riemann surface of genus m also possesses m holomorphic differentials (Farkas, 1967) which we shall here denote by $\{\mathrm{d}v_j(\zeta) | j = 1, ..., m\}$. The functions $\{v_j(\zeta) | j = 1, ..., m\}$ are the integrals of the first kind and each is uniquely determined, up to an additive constant, by its periods around the a and b-cycles. The functions $\{v_j(\zeta) | j = 1, ..., m\}$ are analytic, but not single valued, in F. Here we normalize the holomorphic differentials so that

$$\oint_{a_k} \mathrm{d}v_j = \delta_{jk}, \quad \oint_{b_k} \mathrm{d}v_j = \tau_{jk}. \tag{4.5}$$

The value δ_{jk} is the Kronecker delta, and the matrix of constants τ_{jk} defines the so-called period matrix (Crowdy & Marshall, 2007a; Crowdy, 2010b).

Armed with a normalized basis of a and b-cycles, the m integrals of the first kind and the Schottky group Θ, we have all the necessary machinery to be able to define the S–K prime function. The following theorem is established in Hejhal (1972):

THEOREM 4.1 There is a unique function $X(\zeta, \gamma)$ defined by the properties:

(i) $X(\zeta, \gamma)$ is analytic everywhere in F (except possibly at the point at infinity when it is included in F).

(ii) For $\gamma \in F$, $X(\zeta,\gamma)$ has a second-order zero at each of the points $\{\theta(\gamma)|\theta \in \Theta\}$.

(iii) For $\gamma \in F$,

$$\lim_{\zeta\to\gamma} \frac{X(\zeta,\gamma)}{(\zeta-\gamma)^2} = 1. \tag{4.6}$$

(iv) For $j = 1, ..., m$,

$$X(\theta_j(\zeta),\gamma) = H_j(\zeta,\gamma)X(\zeta,\gamma), \tag{4.7}$$

where

$$H_j(\zeta,\gamma) = \exp\left[-4\pi \mathrm{i}\left(v_j(\zeta) - v_j(\gamma) + \tfrac{1}{2}\tau_{jj}\right)\right] \frac{\mathrm{d}\theta_j(\zeta)}{\mathrm{d}\zeta}. \tag{4.8}$$

Hejhal (1972) then defines the 'Klein prime function' $\omega(\zeta,\gamma)$ (or what we will call, following Baker (1897), the S–K prime function) as the square root of this function, i.e.

$$\omega(\zeta,\gamma) = (X(\zeta,\gamma))^{1/2}, \tag{4.9}$$

where the branch of the square root is chosen so that $\omega(\zeta,\gamma)$ behaves like $(\zeta-\gamma)$ as $\zeta \to \gamma$.

The prime function in multiply connected domains satisfies properties similar to $(\zeta-\gamma)$ relevant in the simply connected setting. First, it satisfies the skew-symmetry relation

$$\omega(\gamma,\zeta) = -\omega(\zeta,\gamma). \tag{4.10}$$

Moreover, the prime function $\omega(\zeta,\gamma)$ associated with multiply connected circular domains has an additional symmetry property (first demonstrated in Crowdy & Marshall (2005) using an infinite product representation and then established more generally in Vasconcelos *et al.* (2014)), given by

$$\overline{\omega}(1/\zeta,1/\gamma) \equiv \overline{\omega(1/\overline{\zeta},1/\overline{\gamma})} = -\frac{1}{\zeta\gamma}\omega(\zeta,\gamma). \tag{4.11}$$

5. Potential theory

Following Crowdy & Marshall (2007a), who derived a forebear of the numerical construction to be presented here, we seek to compute the prime function by identifying a boundary value problem satisfied by $X(\zeta,\alpha)$. The idea here is to show that $X(\zeta,\alpha)$ can be found once some basic potential theoretic functions have been determined, namely the m first kind integrals $\{v_j|j = 1, ..., m\}$ together with a so-called modified Green's function G_j (for some choice of $j = 0, 1, ..., m$). The definition of these basic objects of potential theory are now given in this section. Finding these functions numerically turns out to involve nothing more than the solution of standard boundary value problems for single-valued analytic functions with imaginary parts specified on the boundary components. These are known as modified Schwarz problems (Crowdy, 2008b).

5.1. *First kind integrals* $\{v_j | j = 1, ..., m\}$

It was shown in Crowdy & Marshall (2007a) that, for Schottky doubles of planar multiply connected domains, the integrals of the first kind have some special properties that facilitate their numerical computation. For $j \in \{1, \ldots, m\}$, we may write

$$v_j(\zeta) = \hat{v}_j(\zeta) + \frac{1}{2\pi \mathrm{i}} \begin{cases} \log\left(\dfrac{\zeta - \delta_j}{\zeta - \delta_j'}\right) & |\delta_j| > q_j, \\ \log(\zeta - \delta_j) & 0 \le |\delta_j| \le q_j, \end{cases} \tag{5.1}$$

where $\hat{v}_j$ is single valued in F. These functions satisfy

$$\mathrm{Im}\left[v_j(\zeta)\right] = \begin{cases} \gamma_{j0} = 0 & \zeta \in C_0, \\ \gamma_{jk} & \zeta \in C_k,\ k \in \{1, \ldots, m\}, \end{cases} \tag{5.2}$$

where the set $\{\gamma_{jk} | j, k = 1, ..., m\}$ are real constants which must be found as part of the solution. Using the case $|\delta_j| > q_j$ for purposes of exposition, the modified Schwarz problem for v_j is then defined by

$$\mathrm{Im}\left[\hat{v}_j(\zeta)\right] = \gamma_{jk} + \frac{1}{2\pi} \log\left|\frac{\zeta - \delta_j}{\zeta - \delta_j'}\right| \quad \text{for } \zeta \in C_k. \tag{5.3}$$

The statement for the case $0 \le |\delta_j| \le q_j$ follows in an obvious way.

Another useful property of the first kind integrals (Crowdy & Marshall, 2007a) is

$$\overline{v_j}(1/\zeta) = v_j(\zeta) \quad \text{for all } \zeta \in F. \tag{5.4}$$

Also, the property (Crowdy & Marshall, 2007b)

$$\tau_{jj} = v_j(\theta_j(\zeta)) - v_j(\zeta) \tag{5.5}$$

holds for all points ζ on C_j' and will be needed in what follows.

5.2. *The modified Green's functions*

Suppose α is a point strictly inside D_ζ. Then, as first shown in Crowdy & Marshall (2006, 2007b), the so-called modified Green's functions of D_ζ for $j \in \{0, \ldots, m\}$ can be written in terms of the prime function as

$$G_j(\zeta, \alpha) = \frac{1}{4\pi \mathrm{i}} \log\left[\frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi_j(\alpha)})\overline{\omega}(\phi_j(\zeta), \overline{\alpha})}\right] = \frac{1}{2\pi \mathrm{i}} \log\left[\frac{q_j}{|\alpha - \delta_j|} \frac{\omega(\zeta, \alpha)}{\omega(\zeta, \theta_j(1/\overline{\alpha}))}\right]. \tag{5.6}$$

In Crowdy & Marshall (2007b), these functions were shown to satisfy the boundary conditions

$$\mathrm{Im}\left[G_j(\zeta, \alpha)\right] = \begin{cases} \Gamma_{jj} = 0 & \zeta \in C_j, \\ \Gamma_{jk} & \zeta \in C_k,\ k \ne j, \end{cases} \tag{5.7}$$

such that $\{\Gamma_{jk}\}$ is a set of constants determined by requiring the additional constraints

$$\oint_{C_k} \frac{\partial G_j}{\partial n}\, \mathrm{d}s = 0 \quad k \neq j, \tag{5.8}$$

where $\partial/\partial n$ is the outward normal derivative and ds is the arclength differential. The functions $\{G_j\}$ are all analytic in D_ζ, except for a logarithmic singularity at α, and we may therefore decompose them in the form

$$G_j(\zeta,\alpha) = \hat{G}_j(\zeta,\alpha) + \frac{1}{2\pi \mathrm{i}} \log\left[\frac{\zeta-\alpha}{\zeta-\theta_j(1/\overline{\alpha})}\right], \tag{5.9}$$

where $\hat{G}_j(\zeta,\alpha)$ is analytic in D_ζ. The boundary conditions (5.7) for the G_j then produce a modified Schwarz problem for the single-valued analytic function $\hat{G}_j(\zeta,\alpha)$:

$$\mathrm{Im}\left[\hat{G}_j(\zeta,\alpha)\right] = \Gamma_{jk} + \frac{1}{2\pi}\log\left|\frac{\zeta-\alpha}{\zeta-\theta_j(1/\overline{\alpha})}\right| \quad \text{for } \zeta \in C_k, \ \ k = 0, 1, ..., m. \tag{5.10}$$

Recall that the circle C_0 is not a boundary of F—rather, it is the circle separating the two sides of the Schottky double—which should be a clue that $G_0(\zeta,\alpha)$ behaves slightly differently to the functions $\{G_j(\zeta,\alpha)|j = 1, ..., m\}$. Indeed $G_0(\zeta,\alpha)$ has two logarithmic singularities in the closure of F, at α and at $1/\overline{\alpha}$, with of course the exception of the case where $|\alpha| = 1$, and we single it out by writing

$$G_0(\zeta,\alpha) = \hat{G}_0(\zeta,\alpha) + \frac{1}{2\pi \mathrm{i}}\begin{cases} \log(\zeta) & \alpha = 0, \\ \log\left(\dfrac{\zeta-\alpha}{\zeta-1/\overline{\alpha}}\right) & 0 < |\alpha| < \infty, \\ \log(1/\zeta) & \alpha = \infty. \end{cases} \tag{5.11}$$

It can in fact be shown that any G_j, for $j > 0$, can be written as a combination of the function G_0 and the first kind integral v_j. We thus have the following relationship between G_0 and $\{G_j|j = 1, \ldots, m\}$, which will be useful in the proof of one of the main theorems of this article.

LEMMA 5.1 The modified Green's function $G_j(\zeta,\alpha)$, for $j \in \{1, \ldots, m\}$, can be written in terms of $G_0(\zeta,\alpha)$ and the first-kind integral $v_j(\zeta)$. That is,

$$G_j(\zeta,\alpha) = G_0(\zeta,\alpha) - v_j(\zeta) + \overline{v_j(\alpha)} + \tfrac{1}{2}\tau_{jj} + \frac{1}{2\pi}\arg\left[\frac{\alpha}{\alpha-\delta_j}\right]. \tag{5.12}$$

Moreover, if $\zeta \in C_j$, then

$$G_j(\zeta,\alpha) = \mathrm{Re}\left[G_0(\zeta,\alpha) - v_j(\zeta) + v_j(\alpha)\right] + \frac{1}{2\pi}\arg\left[\frac{\alpha}{\alpha-\delta_j}\right]. \tag{5.13}$$

Proof. Begin with the definition (5.6) of G_j and the transformation property in (4.7). Then by calculation, for any ζ in F, we have

$$\begin{aligned} G_j(\zeta,\alpha) &= \frac{1}{4\pi\mathrm{i}}\log\left[\frac{q_j^2}{|\alpha-\delta_j|^2}\frac{X(\zeta,\alpha)}{X(\zeta,\theta_j(1/\overline{\alpha}))}\right] \\ &= \frac{1}{4\pi\mathrm{i}}\log\left[\frac{X(\zeta,\alpha)}{|\alpha|^2X(\zeta,1/\overline{\alpha})}\right]-v_j(\zeta)+\overline{v_j(\alpha)}+\frac{1}{2}\tau_{jj}+\frac{1}{4\pi\mathrm{i}}\log\left[\frac{\overline{\alpha}-\overline{\delta_j}\,\alpha}{\alpha-\delta_j\,\overline{\alpha}}\right], \end{aligned} \tag{5.14}$$

which proves (5.12).

Now suppose ζ is restricted to circle C_j. From Crowdy & Marshall (2007b) we have $\Gamma_{0j}=\mathrm{Im}\left[v_j(\alpha)\right]$. This, (5.5) and the fact that $\theta_j(1/\overline{\zeta})=\zeta$ when $\zeta\in C_j$, show that (5.12) then becomes (5.13). □

6. The computation of the prime function

To formulate the boundary value problem for $X=\omega^2$ we write

$$\log X(\zeta,\alpha)=\log\hat{X}(\zeta,\alpha)+\begin{cases}\log\,(\zeta-\alpha)^2 & 0\le|\alpha|<\infty,\\ 0 & \alpha=\infty,\end{cases} \tag{6.1}$$

where $\hat{X}$ is analytic in F. We will compute the unknown function $\hat{X}$. It turns out that two separate cases must be considered depending on the location of the point α:

(a) The case where the point α is not on any of the boundary circles $\{C_j|j=0,1,...,m\}$ (Theorem 6.1).

(b) The case where α is located on a boundary circle (Theorem 6.2).

The proofs of these two results are given in Appendices A and B, respectively.

THEOREM 6.1 Suppose α is in the fundamental domain and not on the unit circle, specifically $\alpha\notin\{\partial D_\zeta\cup\partial D'_\zeta\}$. Then $\hat{X}(\zeta,\alpha)$ is an analytic function for $\zeta\in F$, and

$$\mathrm{Im}\left[\log\hat{X}(\zeta,\alpha)\right]=\begin{cases}2\pi\ \mathrm{Re}\left[\hat{G}_0(\zeta,\alpha)\right] & \zeta\in C_0,\\ 2\pi\ \mathrm{Re}\left[\hat{G}_0(\zeta,\alpha)-\hat{v}_j(\zeta)+v_j(\alpha)\right]+A_j(\zeta,\alpha) & \zeta\in C_j,\ j>0,\end{cases} \tag{6.2}$$

where

$$A_j(\zeta,\alpha)=\begin{cases}\arg\left[\dfrac{\zeta-\delta_j'}{\zeta}\right] & q_j<|\delta_j|,\ \alpha\in\{0,\infty\},\\ \arg\left[\dfrac{\alpha(\zeta-\delta_j')}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\right] & q_j<|\delta_j|,\ 0<|\alpha|<\infty,\\ \arg\left[\dfrac{\alpha}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\right] & 0\le|\delta_j|\le q_j.\end{cases} \tag{6.3}$$

REMARK 6.1 Our preference for stating the boundary value problem for $\hat{X}(\zeta,\alpha)$ in terms of G_0 and $\{v_j | j = 1,\ldots,m\}$ is a matter of implementation efficiency. Since the v_j do not depend on the location of α, they may be computed once the domain is set and used repeatedly for any α given.

THEOREM 6.2 Suppose α is located on one of the boundary circles of D_ζ, i.e. $\alpha \in C_k$ given some $k \in \{0,\ldots,m\}$. Then $\hat{X}(\zeta,\alpha)$ is an analytic function for all ζ in F, and

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = \begin{cases} 0 & \zeta \in C_k, \\ 2\pi \operatorname{Re}\left[\hat{v}_k(\zeta) - v_k(\alpha)\right] + A_{0k}(\zeta,\alpha) & \zeta \in C_0,\ k > 0, \\ 2\pi \operatorname{Re}\left[v_j(\alpha) - \hat{v}_j(\zeta)\right] + A_{j0}(\zeta,\alpha) & \zeta \in C_j,\ j \neq k,\ k = 0, \\ 2\pi \operatorname{Re}\left[\hat{v}_k(\zeta) - \hat{v}_j(\zeta) + v_j(\alpha) \right. & \\ \qquad\qquad \left. - v_k(\alpha)\right] + A_{jk}(\zeta,\alpha) & \zeta \in C_j,\ j \neq k,\ j > 0,\ k > 0, \end{cases} \tag{6.4}$$

with

$$A_{0k}(\zeta,\alpha) = \begin{cases} \arg\left[\dfrac{-\delta_k(\zeta-\delta_k)}{\zeta}\right] & |\delta_k| = q_k,\ \alpha = 0, \\ \arg\left[\dfrac{(\alpha-\delta_k)\zeta(\zeta-\delta_k)}{(\zeta-\alpha)^2}\right] & 0 \leq \delta_k \leq q_k,\ \alpha \neq 0, \\ \arg\left[\dfrac{(\alpha-\delta_k)\zeta(\zeta-\delta_k)}{(\zeta-\alpha)^2(\zeta-\delta_k')}\right] & q_k < |\delta_k|, \end{cases} \tag{6.5}$$

where also

$$A_{j0}(\zeta,\alpha) = \begin{cases} \arg\left[\dfrac{\alpha}{(\zeta-\alpha)^2}\right] & 0 \leq |\delta_j| \leq q_j, \\ \arg\left[\dfrac{\alpha(\zeta-\delta_j')}{(\zeta-\alpha)^2}\right] & q_j < |\delta_j|, \end{cases} \tag{6.6}$$

and

$$A_{jk}(\zeta,\alpha) = \begin{cases} \arg\left[\dfrac{-\delta_k(\zeta-\delta_j')(\zeta-\delta_k)}{\zeta^2}\right] & |\delta_k| = q_k,\ \alpha = 0, \\ \arg\left[\dfrac{(\alpha-\delta_k)(\zeta-\delta_j')(\zeta-\delta_k)}{(\zeta-\alpha)^2}\right] & 0 \leq \delta_k \leq q_k,\ q_j < |\delta_j|,\ \alpha \neq 0, \\ \arg\left[\dfrac{(\alpha-\delta_k)(\zeta-\delta_k)}{(\zeta-\alpha)^2(\zeta-\delta_k')}\right] & 0 \leq \delta_j \leq q_j,\ q_k < |\delta_k|, \\ \arg\left[\dfrac{(\alpha-\delta_k)(\zeta-\delta_j')(\zeta-\delta_k)}{(\zeta-\alpha)^2(\zeta-\delta_k')}\right] & q_k < |\delta_k|, q_j < |\delta_j|. \end{cases} \tag{6.7}$$

REMARK 6.2 When α is on an outer boundary circle, $\alpha \in \partial D_\zeta' \setminus C_0$, we may take advantage of the transformation property (4.7). This is discussed further in Appendix C.2.

REMARK 6.3 In all cases above, the solution for $\hat{X}$ will be determined up to a complex constant that will be set by imposing the normalization condition

$$\lim_{\zeta\to\alpha} \frac{(\zeta-\alpha)^2\hat{X}(\zeta,\alpha)}{(\zeta-\alpha)^2} = \hat{X}(\alpha,\alpha) = 1 \tag{6.8}$$

stemming from (4.6).

In the next two sections we present two different numerical approaches to finding $\hat{X}(\zeta,\alpha)$. A traditional numerical approach to solving a modified Schwarz problem would be to employ standard boundary integral methods. However, we do not do that here and instead we present two methods each having more novel mathematical features: the method presented in Section 7 is based on a new transform approach to boundary value problems for analytic functions in circular domains recently formulated in Crowdy (2015a,b); the second method, described in Section 8, is closer to a more traditional boundary integral formulation but nevertheless is unusual in relying on use of the so-called 'generalized Neumann kernel' (Nasser, 2009) rather than the usual Cauchy kernel. In Section 9 we then compare the performance and accuracy of the two schemes.

7. Implementation 1: a spectral method based on global relations

In recent work (Crowdy, 2015a,b) a new transform approach to the solution of boundary value problems for Laplace's equation in multiply connected circular domains was expounded. At the heart of the method is the analysis of the so-called 'global relations', which can be used to determine unknown data associated with analytic functions defined in multiply connected circular domains (including those with straight line edges or polygons). Since the domains D_ζ of interest in this article are all circular it is possible to use the ideas of Crowdy (2015a,b) to find solutions of the modified Schwarz problems needed to compute the S-K prime function. We now show how to do this. Readers may note that we have made available (Kropf, 2015) a downloadable implementation of this approach prepared on the MATLAB platform.

Suppose that on each boundary circle of ∂D_ζ, the boundary values of a single-valued function f analytic in D_ζ are given by real functions ϕ_j and r_j as

$$f(\zeta) = \phi_j(\zeta) + \mathrm{i}r_j(\zeta) + \mathrm{i}\gamma_j \quad \text{for } \zeta \in C_j, \tag{7.1}$$

where $\{r_j(\zeta)\}$ is a set of arbitrary given functions while the constants $\{\gamma_j\}$, and the real parts $\{\phi_j(\zeta)\}$, must be chosen in order to ensure that $f(\zeta)$ is single valued in D_ζ. This is a so-called modified Schwarz problem for $f(\zeta)$.

The approach advocated in this section is simple: the so-called global relations described in Crowdy (2015a,b) are used to find the unknown boundary data associated with the functions $\{v_j | j = 1, ..., m\}$ and, say, the modified Green's function G_0, all of which satisfy modified Schwarz problems of this type. By virtue of Theorems 6.1 and 6.2 the complete set of boundary data for these functions provides the boundary data for the modified Schwarz problem for the required function $\log\hat{X}(\zeta,\alpha)$.

7.1. *Global relations*

Given $j, p \in \{0, 1, \ldots, m\}$ and integers k, consider the integral functions

$$L_{p,j}(f;k) = \sigma(j) \begin{cases} \oint_{C_j} f(\zeta)\zeta^{-k-1}\,\mathrm{d}\zeta & \text{if } p = 0, \\ \oint_{C_j} f(\zeta)\left(\frac{\zeta-\delta_p}{q_p}\right)^k \mathrm{d}\zeta & \text{if } p > 0, \end{cases} \tag{7.2}$$

where integration is anti-clockwise on each circle and

$$\sigma(j) = \begin{cases} +1 & j = 0, \\ -1 & j > 0. \end{cases} \tag{7.3}$$

The following theorem sets out the global relations, which encode the analyticity of f in D_ζ:

THEOREM 7.1 If f is a single-valued, analytic function in D_ζ, then

$$\sum_{j=0}^{m} L_{p,j}(f;k) = 0 \tag{7.4}$$

holds for all $0 \le p \le m$ and all integers $k < 0$.

Proof. The proof follows from the fact that, for z in the complement of the closure of D_ζ, Cauchy's theorem states

$$\int_{\partial D_\zeta} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0. \tag{7.5}$$

We first suppose $|z| > 1$. Then

$$0 = \int_{\partial D_\zeta} -\frac{1}{z}\frac{f(\zeta)}{1-\zeta/z}\,\mathrm{d}\zeta = -\frac{1}{z}\int_{\partial D_\zeta} f(\zeta)\sum_{k=0}^{\infty}\left(\frac{\zeta}{z}\right)^k \mathrm{d}\zeta = -\sum_{k=0}^{\infty} z^{-k-1}\int_{\partial D_\zeta} f(\zeta)\zeta^k\,\mathrm{d}\zeta. \tag{7.6}$$

But this means that the integral on the right-hand side is zero for all $k \ge 0$ or equivalently,

$$\int_{\partial D_\zeta} f(\zeta)\zeta^{-k-1}\,\mathrm{d}\zeta = 0 \quad \text{for all } k < 0. \tag{7.7}$$

Now suppose $|z - \delta_j| < q_j$ for some $1 \le j \le m$. Then

$$\begin{aligned} 0 &= \int_{\partial D_\zeta} \frac{f(\zeta)}{\zeta - \delta_j}\frac{1}{1-(z-\delta_j)/(\zeta-\delta_j)}\,\mathrm{d}\zeta \\ &= \int_{\partial D_\zeta} \frac{f(\zeta)}{\zeta-\delta_j}\sum_{k\ge 0}\left(\frac{z-\delta_j}{\zeta-\delta_j}\right)^k \mathrm{d}\zeta = \sum_{k\ge 0}\frac{(z-\delta_j)^k}{q_j^{k+1}}\int_{\partial D_\zeta} f(\zeta)\left(\frac{\zeta-\delta_j}{q_j}\right)^{-k-1}\mathrm{d}\zeta, \end{aligned} \tag{7.8}$$

but this means that the integral on the right is zero for all $k \geq 0$, or equivalently

$$\int_{\partial D_\zeta} f(\zeta) \left(\frac{\zeta - \delta_j}{q_j} \right)^k \mathrm{d}\zeta = 0 \quad \text{for all } k < 0. \tag{7.9}$$

The value of the sum then follows directly from the definition of $L_{p,j}$. □

REMARK 7.1 The global relations (7.4) are identical to those given in Crowdy (2015a,b) where they were derived in the context of certain transform pairs associated with finding analytic functions defined over general circular domains.

7.2. *Forming a linear system*

Define the linear function

$$\eta_j(\zeta) := \frac{\zeta - \delta_j}{q_j}, \tag{7.10}$$

so that for $\zeta \in C_j$ parameterized by $\zeta = \delta_j + q_j \mathrm{e}^{\mathrm{i}\theta}$, we have $\eta_j(\zeta) = \mathrm{e}^{\mathrm{i}\theta}$. For each unknown function $\phi_j(\zeta) + \mathrm{i}\gamma_j$, we may write the decomposition

$$\phi_j(\zeta) + \mathrm{i}\gamma_j = a_{j,0} + \sum_{n \geq 1} \left(a_{j,n} \eta_j^n(\zeta) + \overline{a_{j,n}} \eta_j^{-n}(\zeta) \right) \quad \text{for } \zeta \in C_j, \tag{7.11}$$

where we take $\gamma_j = \mathrm{Im}\left[a_{j,0}\right]$. From the boundary value problem it is clear that f will only be determined up to a real additive constant, so without loss of generality we take $a_{0,0} = 0$. Applying the global relations in Theorem 7.1 we use the above form of f on each C_j to see that

$$\sum_{j=0}^{m} L_{p,j}(\phi_j + \mathrm{i}\gamma_j; k) = -\sum_{j=0}^{m} L_{p,j}(\mathrm{i}r_j; k) \tag{7.12}$$

for $p \in \{0, \ldots, m\}$ and all $k < 0$. Substituting the series expansions into the L integrals along with the parameterized ζ values, we find that

$$\begin{aligned} L_{0,j}(\phi_j + \mathrm{i}\gamma_j; k) = \mathrm{i}q_j\sigma(j) \Bigg\{ & a_{j,0} \int_0^{2\pi} (\delta_j + q_j \mathrm{e}^{\mathrm{i}\theta})^{-k-1} \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta \\ & + \sum_{n \geq 1} a_{j,n} \int_0^{2\pi} (\delta_j + q_j \mathrm{e}^{\mathrm{i}\theta})^{-k-1} \mathrm{e}^{\mathrm{i}(1+n)\theta} \, \mathrm{d}\theta \\ & + \sum_{n \geq 1} \overline{a_{j,n}} \int_0^{2\pi} (\delta_j + q_j \mathrm{e}^{\mathrm{i}\theta})^{-k-1} \mathrm{e}^{\mathrm{i}(1-n)\theta} \, \mathrm{d}\theta \Bigg\}, \end{aligned} \tag{7.13}$$

and for $p > 0$

$$L_{p,j}(\phi_j + \mathrm{i}\gamma_j; k) = \mathrm{i}q_j\sigma(j)\left\{a_{j,0}\int_0^{2\pi}\left(\frac{\delta_j-\delta_p}{q_p}+\frac{q_j}{q_p}\mathrm{e}^{\mathrm{i}\theta}\right)^k \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \right.$$
$$+\sum_{n\geq 1} a_{j,n}\int_0^{2\pi}\left(\frac{\delta_j-\delta_p}{q_p}+\frac{q_j}{q_p}\mathrm{e}^{\mathrm{i}\theta}\right)^k \mathrm{e}^{\mathrm{i}(1+n)\theta}\,\mathrm{d}\theta \tag{7.14}$$
$$\left. +\sum_{n\geq 1} \overline{a_{j,n}}\int_0^{2\pi}\left(\frac{\delta_j-\delta_p}{q_p}+\frac{q_j}{q_p}\mathrm{e}^{\mathrm{i}\theta}\right)^k \mathrm{e}^{\mathrm{i}(1-n)\theta}\,\mathrm{d}\theta\right\}.$$

The integrals in these equations all fit the pattern

$$I(K,N) = \int_0^{2\pi}\left(A + B\mathrm{e}^{\mathrm{i}\theta}\right)^K \mathrm{e}^{\mathrm{i}N\theta}\,\mathrm{d}\theta, \tag{7.15}$$

which we will call the 'binomial integral' on the circle. A significant advantage of our approach is that all such integrals can be evaluated in closed form without the need for numerical integration.

Our application involves three cases of the binomial integral that are not difficult to establish. We begin by defining the binomial coefficient in the standard way,

$$\binom{x}{n} := \frac{x(x+1)(x+2)\cdots(x+n-1)}{n!}. \tag{7.16}$$

If one assumes $K \geq 0$, it then follows that

$$I(K,N) = 2\pi\begin{cases} A^{(K+N)}B^{-N}\binom{K}{-N} & -K \leq N \leq 0,\\ 0 & N \leq -K \text{ or } 0 < N.\end{cases} \tag{7.17}$$

In the second case assume $K < 0$ and $|A| > |B|$, and it can be shown that

$$I(K,N) = 2\pi\begin{cases} A^K\left(\frac{B}{A}\right)^{-N}\binom{K}{-N} & N \leq 0,\\ 0 & 0 < N.\end{cases} \tag{7.18}$$

Finally, take $K < 0$ and $|A| < |B|$, which leads us to

$$I(K,N) = 2\pi\begin{cases} B^K\left(\frac{A}{B}\right)^{K+N}\binom{K}{K+N} & -K \leq N,\\ 0 & N < -K.\end{cases} \tag{7.19}$$

For our domains, we never need to consider $|A| = |B|$. We summarize with the following theorem.

THEOREM 7.2 The integrals $L_{p,j}(\phi_j + \mathrm{i}\gamma_j; k)$, given $k < 0$, take on the following values. If $p = 0$ we have

$$L_{0,0}(\phi_0; k) = 2\pi\,\mathrm{i}\overline{a_{0,-k}}, \tag{7.20}$$

or

$$L_{0,j}(\phi_j + \mathrm{i}\gamma_j; k) = -2\pi\mathrm{i}\sum_{n=1}^{-k}\binom{-k-1}{n-1} q_j^n \delta_j^{-k-n}\overline{a_{j,n}} \qquad \text{if } j > 0. \tag{7.21}$$

When $p > 0$ there are three cases:

1. $j = 0$:

$$L_{p,0}(\phi_0; k) = 2\pi\mathrm{i} q_p^{-k} \sum_{n\geq \max\{1,-k-1\}} \binom{k}{k+1+n} (-\delta_p)^{k+1+n} a_{0,n}, \tag{7.22}$$

2. $j > 0, j \neq p$:

$$L_{p,j}(\phi_j + \mathrm{i}\gamma_j; k) = -2\pi\mathrm{i} q_p^{-k} \sum_{n\geq 1} \binom{k}{n-1} q_j^n (\delta_j - \delta_p)^{k+1-n}\overline{a_{j,n}}, \tag{7.23}$$

3. and $j = p$:

$$L_{p,p}(\phi_p + \mathrm{i}\gamma_p; k) = -2\pi\mathrm{i} q_p a_{p,-k-1}. \tag{7.24}$$

Proof. Apply the binomial integral values to (7.13) and (7.14). □

7.3. *Discretization*

We begin by truncating each series $\phi_j + \mathrm{i}\gamma_j$ so there are $1 + 2N$ terms for each j. That is,

$$\tilde{\phi}_j(\zeta) + \mathrm{i}\gamma_j = a_{j,0} + \sum_{n=1}^{N} \left(a_{j,n}\eta_j^n(\zeta) + \overline{a_{j,n}}\eta_j^{-n}(\zeta) \right). \tag{7.25}$$

Let x be a vector of unknowns,

$$x := \left(a_{0,1}; \ldots; a_{0,N}; a_{1,0}; \ldots; a_{1,N}; a_{2,0}; \ldots; a_{m,N} \right). \tag{7.26}$$

Then x has $Q := (m+1)N - 1$ elements. Using the global relations in the form (7.12) we build a system of equations such that

$$L\begin{pmatrix} x \\ \overline{x} \end{pmatrix} = r, \tag{7.27}$$

where the bar over x represents element-wise complex conjugation, and r represents the integrals on the right-hand side of (7.12) appropriately. The matrix L should have $2Q$ columns, and to determine the number of rows let $k \in \{-1, \ldots, -N\}$ if $p = 0$, and $k \in \{-1, \ldots, -N-1\}$ for $1 \leq p \leq m$. Limiting

k when $p = 0$ amounts to dealing with coefficients $\{a_{0,1}; \ldots; a_{0,N}\}$, and otherwise limits the maximum possible coefficient index to N by the equations in Theorem 7.2, which is the desired discretization behaviour. Thus L has Q rows, and r is a column vector with Q elements.

Now write $L = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ where both sub-matrices are Q-by-Q. Then the system of (7.27) implies

$$A_1 x + A_2 \overline{x} = r. \tag{7.28}$$

If we consider the total number of unknowns to be $2Q$, x and $\overline{x}$, then we need Q more complex equations to match the number of unknowns. These are provided by taking conjugates:

$$\overline{A_2} x + \overline{A_1} \overline{x} = \overline{r}, \tag{7.29}$$

where the bar over the matrix indicates element-wise conjugation. The new (square) system is then given by

$$A \begin{pmatrix} x \\ \overline{x} \end{pmatrix} := \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} x \\ \overline{x} \end{pmatrix} = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}. \tag{7.30}$$

The trapezoidal rule is used to compute the vectors r and $\overline{r}$. A basic linear solver, the MATLAB backslash operator, is used to solve the system. The method thus has a computational cost of $O(m^3 N^3)$.

Since each boundary value problem in Section 6 is given in terms of the known imaginary part of an analytic function on ∂D_ζ, it should be clear how to apply (7.30) to find the coefficients in the series expansions of the real part of the function, along with the unknown imaginary constants. The general strategy is summarized as follows:

(1) First apply (7.30) to find the m functions $\{v_j | j = 1, ..., m\}$ from the modified Schwarz problem given in (5.3).

(2) Next apply (7.30) to find the function $\hat{G}_0$, and hence G_0, via the modified Schwarz problem given in (5.10).

(3) Use the results from (1) and (2) above to provide the boundary data needed in Theorems 6.1 and 6.2 to solve the modified Schwarz problem for $\log \hat{X}(\zeta, \alpha)$ and, hence, to determine the boundary values of the S-K prime function.

(4) Use a numerical continuation procedure to find values of $\log \hat{X}(\zeta, \alpha)$, and hence the S-K prime function, inside the domain D_ζ. Some technical details of this procedure are given in Appendix C.

8. Implementation 2: a boundary integral method using the generalized Neumann kernel

In this section Theorems 6.1 and 6.2 are again used to determine the S-K prime function by first determining solutions of modified Schwarz problems for $\{v_j | j = 1, ..., m\}$ and the modified Green's function G_0. This time, however, these functions are found by solving integral equations based on use of the generalized Neumann kernel, which is to be described next. A downloadable MATLAB implementation of this approach is available in Nasser (2016).

8.1. *The generalized Neumann kernel*

Consider the circles constituting the boundary of D_ζ. The exterior circle C_0 is parameterized by

$$\zeta_0(t) = \mathrm{e}^{\mathrm{i}t} \quad 0 \le t \le 2\pi, \tag{8.1}$$

and the inner circles $\{C_j : j = 1, \ldots, m\}$, are parameterized by

$$\zeta_j(t) = \delta_j + q_j \mathrm{e}^{-\mathrm{i}t} \quad 0 \le t \le 2\pi. \tag{8.2}$$

For $j = 0, 1, \ldots, m$, define $J_j := [0, 2\pi]$. Define also J as the disjoint union of $m+1$ intervals $J_0, J_1, \ldots, J_m$,

$$J = \bigsqcup_{j=0}^{m} J_j = \bigcup_{j=0}^{m} \{(t,j) \ : \ t \in J_j\}, \tag{8.3}$$

i.e. the elements of J are ordered pairs (t,j), where j is an auxiliary index indicating which of the intervals contains the point t. We define a parameterization of the whole boundary ∂D_ζ as the complex function ζ defined on the total parameter domain J by

$$\zeta(t,j) = \zeta_j(t), \quad t \in J_j \quad j = 0, 1, \ldots, m. \tag{8.4}$$

We shall assume for a given t that the auxiliary index j is known so we replace the pair (t,j) in the left-hand side of (8.4) by t. Thus, the function ζ in (8.4) is written as

$$\zeta(t) = \begin{cases} \mathrm{e}^{\mathrm{i}t} & t \in J_0, \\ \delta_1 + q_1 \mathrm{e}^{-\mathrm{i}t} & t \in J_1, \\ \quad \vdots & \\ \delta_m + q_m \mathrm{e}^{-\mathrm{i}t} & t \in J_m. \end{cases} \tag{8.5}$$

Here we do not distinguish between the notations $\chi(t)$ and $\chi(\zeta(t))$ since, in view of the parameterization (8.5) of the boundary ∂D_ζ, any real-valued or complex-valued function $\chi(\zeta)$ Hölder continuous on the boundary ∂D_ζ can be interpreted via $\hat{\chi}(t) := \chi(\zeta(t))$ as a 2π-periodic Hölder continuous function of the parameter t on J and vice versa.

We assume that $\hat{\alpha}$ is a given point in D_ζ and define a complex-valued function $A(\zeta)$ on ∂D_ζ by

$$A(\zeta) = \zeta - \hat{\alpha}. \tag{8.6}$$

The generalized Neumann kernel $N(s,t)$ and the kernel $M(s,t)$ formed with the functions $A(t)$ and $\zeta(t)$ are defined on $J \times J$ by (Wegmann & Nasser, 2008)

$$N(s,t) := \frac{1}{\pi} \operatorname{Im}\left(\frac{A(s)}{A(t)} \frac{\dot{\zeta}(t)}{\zeta(t) - \zeta(s)} \right), \tag{8.7}$$

$$M(s,t) := \frac{1}{\pi} \operatorname{Re}\left(\frac{A(s)}{A(t)} \frac{\dot{\zeta}(t)}{\zeta(t) - \zeta(s)} \right), \tag{8.8}$$

where the dot denotes the derivative with respect to the parameter t. The kernel N is continuous and the kernel M has a singularity of cotangent type. Let $\mathbf{N}$ and $\mathbf{M}$ be the integral operators with the kernels N and M, respectively. Then $\mathbf{N}$ is a Fredholm integral operator and the integral operator $\mathbf{M}$ is a singular operator. Then from Nasser (2009) we have the following theorem.

THEOREM 8.1 For a given real function φ, Hölder continuous on ∂D_ζ, there exist unique real functions ψ and h such that

$$Af = \varphi + h + \mathrm{i}\psi \tag{8.9}$$

are boundary values of an analytic function on D_ζ. The function ψ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\psi = -\mathbf{M}\varphi \tag{8.10}$$

and the function h is given by

$$h = [\mathbf{M}\psi - (\mathbf{I} - \mathbf{N})\varphi]/2. \tag{8.11}$$

The function h is a piecewise constant function on the boundary ∂D_ζ, i.e.

$$h(\zeta) = \begin{cases} h_0 & \zeta \in C_0, \\ h_1 & \zeta \in C_1, \\ \;\vdots & \\ h_m & \zeta \in C_m, \end{cases} \tag{8.12}$$

with real constants $h_0, h_1, \ldots, h_m$.

For simplicity, a piecewise constant function on the boundary ∂D_ζ of the form given in (8.12) will be denoted by

$$h(\zeta) = (h_0, h_1, \ldots, h_m). \tag{8.13}$$

The boundary integral equation (8.10) is solved using the MATLAB function `fbie`, presented in Nasser (2015). In this function, the integral equation (8.10) is solved accurately by the Nyström method with the trapezoidal rule (Atkinson, 1997; Kress, 2014). Each interval J_j for $j = 0, 1, \ldots, m$ is discretized by n equidistant nodes

$$s_{j,k} = (k-1)\frac{2\pi}{n} \quad k = 1, 2, \ldots, n. \tag{8.14}$$

The total number of nodes in the parameter domain J is $(m+1)n$. Hence, we obtain an $(m+1)n \times (m+1)n$ linear system which is solved using the MATLAB function `gmres`. Each iteration of the GMRES method (Saad & Schultz, 1986) requires a matrix-vector product, which is computed using the function `zfmm2dpart` in the MATLAB toolbox `FMMLIB2D` developed by Greengard & Gimbutas (2012). In this way the integral equation (8.10) is solved in $O((m+1)n\ln n)$ operations. In the function `fbie`, we choose `iprec` $= 5$ (the tolerance of the FMM is 0.5×10^{-15}), `restart` $= 10$ (the GMRES method is restarted every 10 inner iterations), `gmrestol` $= 10^{-15}$ (the tolerance of the GMRES method is 10^{-15}) and `maxit` $= 10$ (the maximum number of outer iterations of GMRES method is 10). See Nasser (2015) for more details.

The order of the convergence of the Nyström method is based on the order of the convergence of the trapezoidal rule which in turn depends on the smoothness of the integrand. Under suitable regularity assumptions on the integrand, the order of the convergence of the Nyström method is the same as the order of the convergence of the trapezoidal rule (see e.g. Atkinson (1997, p. 109) and Kress (2014, p. 227)). Since the boundaries of the domains considered in this article are circles they are C^∞ smooth. Thus, in view of the definitions of the functions $\varphi_0, \varphi_1, \ldots, \varphi_m$ and $\hat{\varphi}_1$, which will be introduced below, the integrand in the integral equation is C^∞ smooth. Hence, the rate of convergence is $O(e^{-cn})$ for some positive constant c depending on the integrand.

By obtaining approximations to ψ and h, we obtain approximations to the boundary values of the function f. The values of $f(z)$ at interior points $z \in D_\zeta$ can be computed by the numerical continuation method outlined in Appendix C.

8.2. *First kind integrals* $\{v_j | j = 1, ..., m\}$

Let $\hat{v}_j(\hat{\alpha}) = a_j + \mathrm{i}b_j$ and the function $f_j(\zeta)$ be defined by

$$f_j(\zeta) = \frac{\hat{v}_j(\zeta) - a_j - \mathrm{i}b_j}{\mathrm{i}A(\zeta)}. \tag{8.15}$$

Then the function f_j is analytic in the domain D_ζ and its boundary values satisfy

$$\mathrm{Re}[A(\zeta)f_j(\zeta)] = \varphi_j(\zeta) + h_j(\zeta), \tag{8.16}$$

where based on the definition of v_j and (5.3) we define

$$\varphi_j(\zeta) = \frac{1}{2\pi}\begin{cases} \log\left|\frac{\zeta-\delta_j}{\zeta-\delta_j'}\right| & |\delta_j| > q_j, \\ \log\left|\zeta - \delta_j\right| & |\delta_j| \le q_j, \end{cases} \tag{8.17}$$

and $h_j(\zeta) = (h_{j0}, h_{j1}, \ldots, h_{jm})$ is a piecewise constant function

$$h_{jk} = \gamma_{jk} - b_j. \tag{8.18}$$

Let $\psi_j(\zeta) = \mathrm{Im}[A(\zeta)f_j(\zeta)]$, then by Theorem 8.1 the function ψ_j is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\psi_j = -\mathbf{M}\varphi_j \tag{8.19}$$

and the piecewise constant function h_j is given by

$$h_j = [\mathbf{M}\psi_j - (\mathbf{I} - \mathbf{N})\varphi_j]/2. \tag{8.20}$$

Then the function $\hat{v}_j$ is given by

$$\hat{v}_j(\zeta) = \mathrm{i}A(\zeta)f_j(\zeta) + a_j + \mathrm{i}b_j \tag{8.21}$$

and the constants γ_{jk} from the statement of the problem (5.3) for the v_j are given by

$$\gamma_{jk} = h_{jk} + b_j, \quad j = 1, 2, \ldots, m, \quad k = 0, 1, 2, \ldots, m. \tag{8.22}$$

If we assume that $\gamma_{j0} = 0$, then we have $b_j = -h_{j0}$. The real constants $\{a_j \,|\, j = 1, \ldots, m\}$ are still undetermined. But, as will be seen below, we do not need these values in our numerical calculation.

8.3. *The modified Green's function*

Assume that α is in the fundamental domain. Let $\hat{G}_0(\hat{\alpha}, \alpha) = a_0 + \mathrm{i}b_0$ and the function $f_0(\zeta)$ be defined by

$$f_0(\zeta) = \frac{\hat{G}_0(\zeta, \alpha) - (a_0 + \mathrm{i}b_0)}{\mathrm{i}A(\zeta)}. \tag{8.23}$$

Then the function f_0 is analytic in the domain D_ζ and its boundary values satisfy

$$\mathrm{Re}[A(\zeta)f_0(\zeta)] = \varphi_0(\zeta) + h_0(\zeta), \tag{8.24}$$

where, based on (5.10) and the definition of G_0, we define

$$\varphi_0(\zeta) = \frac{1}{2\pi}\begin{cases} \log|\zeta| & \alpha = 0, \\ \log\left|\frac{\zeta-\alpha}{\zeta-1/\bar{\alpha}}\right| & 0 < |\alpha| < \infty, \\ -\log|\zeta| & \alpha = \infty, \end{cases} \tag{8.25}$$

and $h_0(\zeta) = (h_{00}, h_{01}, \ldots, h_{0m})$ is a piecewise constant function with

$$h_{0k} = \Gamma_{0k} - b_0 \quad k = 0, 1, \ldots, m. \tag{8.26}$$

Let $\psi_0(\zeta) = \mathrm{Im}[A(\zeta)f_0(\zeta)]$, then by Theorem 8.1 ψ_0 is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\psi_0 = -\mathbf{M}\varphi_0 \tag{8.27}$$

and the piecewise constant function h_0 is given by

$$h_0 = [\mathbf{M}\psi_0 - (\mathbf{I} - \mathbf{N})\varphi_0]/2. \tag{8.28}$$

Then the analytic part of G_0 is given by

$$\hat{G}_0(\zeta, \alpha) = \mathrm{i}A(\zeta)f(\zeta) + a_0 + \mathrm{i}b_0 \tag{8.29}$$

and the unknown constants Γ_{0k} from the boundary value problem statement (5.10) are given by

$$\Gamma_{0k} = h_{0k} + b_0 \quad k = 0, 1, \ldots, m. \tag{8.30}$$

Since $\Gamma_{00} = 0$, we have $b_0 = -h_{00}$. The real constant a_0 in (8.29) is still undetermined. However, as we will see in the rest of this section, we do not need the value of a_0 for the numerical calculations.

8.4. *The prime function*

Once again we employ the key results of Theorems 6.1 or 6.2 to provide the relevant boundary value problems to be solved for the prime function.

Suppose that $\alpha \in F \setminus \partial D'_\zeta$—i.e. we take α to be any point in F including the circular boundaries $\{C_j | j = 0, 1, , ..., m\}$ but excluding the outer circular boundaries $\{C'_j | j = 1, , ..., m\}$ (the latter will be treated separately). The single-valued analytic function $\log \hat{X}(\zeta, \alpha)$ is a solution of the modified Schwarz problem

$$\mathrm{Im}[\log \hat{X}(\zeta, \alpha)] = \hat{\varphi}_1(\zeta) + \hat{h}_1(\zeta), \tag{8.31}$$

where, based on Theorems 6.1 or 6.2, the function $\hat{\varphi}_1(\zeta)$ is given for $\alpha \in D_\zeta \cup D'_\zeta$ by

$$\hat{\varphi}_1(\zeta) = -2\pi\psi_0(\zeta) + \begin{cases} A_0(\zeta, \alpha) & \zeta \in C_0, \\ 2\pi\psi_j(\zeta) + A_j(\zeta, \alpha) & \zeta \in C_j, \end{cases} \tag{8.32}$$

for $\alpha \in C_0$ by

$$\hat{\varphi}_1(\zeta) = \begin{cases} 0 & \zeta \in C_0, \\ \arg\left(\frac{\zeta - \delta'_j}{(\zeta-\alpha)^2}\right) + 2\pi\psi_j(\zeta) & \zeta \in C_j, \end{cases} \tag{8.33}$$

and for $\alpha \in C_k$, $k = 1, 2, \ldots, m$, by

$$\hat{\varphi}_1(\zeta) = \begin{cases} -2\pi\psi_k(\zeta) + A_{0k}(\zeta, \alpha) & \zeta \in C_0, \\ 0 & \zeta \in C_k, \\ -2\pi\psi_k(\zeta) + 2\pi\psi_j(\zeta) + A_{jk}(\zeta, \alpha) & \zeta \in C_j,\ j \neq k. \end{cases} \tag{8.34}$$

The function $\hat{h}_1(\zeta)$ is an unknown piecewise constant function. Let the function $\hat{f}_1(\zeta)$ be defined by

$$\hat{f}_1(\zeta) = \frac{\log \hat{X}(\zeta, \alpha) - \log \hat{X}(\hat{\alpha}, \alpha)}{\mathrm{i}A(\zeta)}. \tag{8.35}$$

It is analytic in the domain D_ζ and its boundary values satisfy

$$\mathrm{Re}[A(\zeta)\hat{f}_1(\zeta)] = \hat{\varphi}_1(\zeta) + \hat{h}_1(\zeta). \tag{8.36}$$

Let $\hat{\psi}_1(\zeta) = \mathrm{Im}[A(\zeta)f(\zeta)]$, then by Theorem 8.1 the function $\hat{\psi}_1$ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\hat{\psi}_1 = -\mathbf{M}\hat{\varphi}_1 \tag{8.37}$$

and the piecewise constant function $\hat{h}_1$ is given by

$$\hat{h}_1 = [\mathbf{M}\hat{\psi}_1 - (\mathbf{I} - \mathbf{N})\hat{\varphi}_1]/2. \tag{8.38}$$

The function $\log \hat{X}(\zeta, \alpha)$ is then given by

$$\log \hat{X}(\zeta, \alpha) = \mathrm{i}A(\zeta)\hat{f}_1(\zeta) + \log \hat{X}(\hat{\alpha}, \alpha). \tag{8.39}$$

By (6.1), $X(\zeta, \alpha)$ can be written as

$$X(\zeta, \alpha) = \begin{cases} \hat{X}(\hat{\alpha}, \alpha)(\zeta - \alpha)^2 \,\mathrm{e}^{\mathrm{i}(\zeta - \hat{\alpha})\hat{f}_1(\zeta)} & \alpha \neq \infty, \\ -\hat{X}(\hat{\alpha}, \alpha)\,\mathrm{e}^{\mathrm{i}(\zeta - \hat{\alpha})\hat{f}_1(\zeta)}/\zeta^2 & \alpha = \infty, \end{cases} \tag{8.40}$$

and $\omega(\zeta, \alpha)$ is given for $\zeta \in \overline{D_\zeta}$ by

$$\omega(\zeta, \alpha) = \begin{cases} c(\zeta - \alpha)\,\mathrm{e}^{\mathrm{i}(\zeta - \hat{\alpha})\hat{f}_1(\zeta)/2} & \alpha \neq \infty, \\ c\,\mathrm{e}^{\mathrm{i}(\zeta - \hat{\alpha})\hat{f}_1(\zeta)/2}/\zeta & \alpha = \infty, \end{cases} \tag{8.41}$$

where c is an undetermined constant. Calculating c is easy for α in the closure of D_ζ, since by the normalization condition (4.6) and our choice of the square root branch

$$\lim_{\zeta \to \alpha} \frac{\omega(\zeta, \alpha)}{\zeta - \alpha} = 1, \tag{8.42}$$

and hence we have

$$c = \lim_{\zeta \to \alpha} \left(\frac{\omega(\zeta, \alpha)}{\zeta - \alpha}\,\mathrm{e}^{-\mathrm{i}(\zeta - \hat{\alpha})\hat{f}_1(\zeta)/2} \right) = \mathrm{e}^{-\mathrm{i}(\alpha - \hat{\alpha})\hat{f}_1(\alpha)/2}, \tag{8.43}$$

where $\hat{f}_1(\alpha)$ is known.

For the parameter $\alpha \in D'_\zeta$, finding the unknown constant c requires extra calculations. On substituting $\zeta = 1$ in (8.41), the constant c is given by

$$c = \begin{cases} \omega(1, \alpha)\,\mathrm{e}^{-\mathrm{i}(1 - \hat{\alpha})\hat{f}_1(1)/2}/(1 - \alpha) & \alpha \neq \infty, \\ \omega(1, \alpha)\,\mathrm{e}^{-\mathrm{i}(1 - \hat{\alpha})\hat{f}_1(1)/2} & \alpha = \infty, \end{cases} \tag{8.44}$$

where $\hat{f}_1(1)$ is known and $\omega(1, \alpha)$ still unknown. However, the value of $\omega(1, \alpha)$ can be computed from

$$\omega(1, \alpha) = \begin{cases} -\alpha\,\overline{\omega(1, 1/\overline{\alpha})} & \alpha \neq \infty, \\ \overline{\omega(1, 0)} & \alpha = \infty, \end{cases} \tag{8.45}$$

where the values on the right-hand side can be computed as explained above since $1/\overline{\alpha} \in D_\zeta$ for $\alpha \neq \infty$ and $0 \in D_\zeta$ for $\alpha = \infty$.

It only remains to account for the case where the parameter α is on an outer circle boundary ($\alpha \in \partial D'_\zeta$). For this we use the transformation property (4.7) and the fact (4.11) that there is a functional relationship involving points in D_ζ and points in D'_ζ. Details are given in Appendix C.

To continue the solution of the boundary value problem outside the unit disc, we simply need to use the conjugate property of the prime function given by (4.11). Since $1/\overline{\zeta}$ is in the closure of D_ζ, the values for $\omega(1/\overline{\zeta}, 1/\overline{\alpha})$ are computed as above. See again Appendix C.

The computational cost for this method is given by noting the function `fbie` is called $m+2$ times to compute the functions $\psi_0, \psi_1, \ldots, \psi_m$, and $\hat{\psi}_1$. If $\alpha \in D'_\zeta$, then two more applications of the function `fbie` are required for the value $\omega(1, \alpha)$.

9. Comparison and performance

In this section we study the numerical characteristics of the two implementations just described. For brevity, we will refer to the implementation presented in Section 7 as Method 1 (M1), and the implementation in Section 8 as Method 2 (M2).

9.1. *Accuracy*

To test the accuracy of M1 and M2 we use a formula given in Crowdy (2008b) for solving the modified Schwarz problem in terms of the prime function. Let $f_s(\zeta)$ be a single-valued analytic function in D_ζ, then

$$f_s(\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D_\zeta} \mathrm{Re}\left[f_s(\zeta)\right] \left(\mathrm{d}\log \omega(\zeta, \alpha) + \mathrm{d}\log \overline{\omega}(\overline{\zeta}, 1/\alpha)\right) + \mathrm{i}C, \tag{9.1}$$

where C is an arbitrary real constant. It should be emphasized that this formula cannot, of course, be used to solve the various modified Schwarz problems arising in this article because it is precisely the function $\omega(\zeta, \alpha)$ that we seek to find! But with $\omega(\zeta, \alpha)$ computed using M1 or M2, and by picking test functions $f_s(\zeta)$, the integral (9.1) can be used to test the accuracy of the computational methods. Let our test functions be

$$f_1(\zeta) = \mathrm{e}^{\zeta}, \tag{9.2}$$

$$f_2(\zeta) = \sum_{j=1}^{m} \left[\frac{1}{(z-\delta_j)} + \frac{1}{(z-\delta'_j)}\right], \tag{9.3}$$

$$f_3(\zeta) = \frac{(z - 0.087 + 0.202\mathrm{i})(z - 0.54 - 0.22\mathrm{i})}{(z + 2 - 1.5\mathrm{i})(z - 2 - \mathrm{i})}. \tag{9.4}$$

These were substituted into both sides of (9.1), with the S-K prime function on the right-hand side computed using M1 and M2, and the results compared at various points in the domain to find a maximum relative error value. Graphs of this relative error for the two methods are shown in Fig. 2.

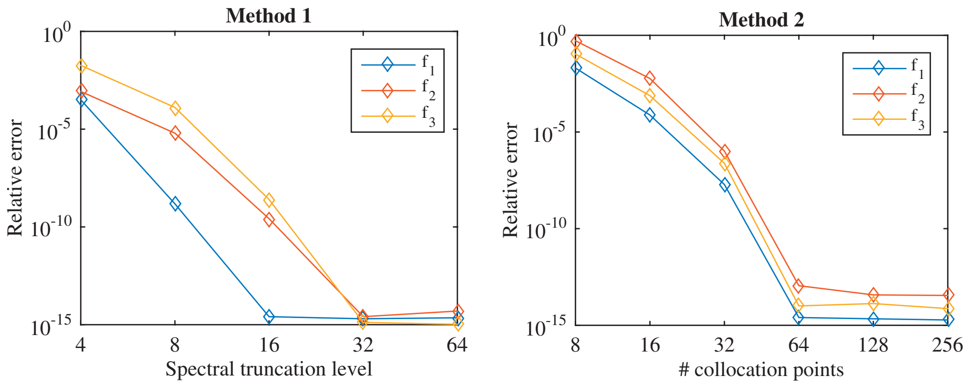

FIG. 2. Relative errors for M1 and M2 based on use of (9.1) as a consistency check.

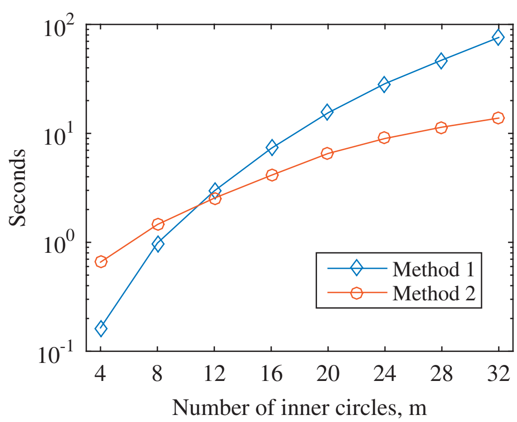

FIG. 3. Run time in seconds for each method with domains of varying connectivity m. The advantages of M2 over M1 become apparent for larger m.

9.2. *Timing*

It is instructive to compare the timing of the two methods. To do so we generate random circular domains for each value of connectivity m with a uniform inner radius size $q = \sqrt{0.15/m}$ and minimum circle separation q. The parameter α for each domain is also chosen randomly and is at least distance q from any boundary. The solution to the boundary value problem for $\hat{X}$ is then computed for each domain. Note that M1 implemented with a generic matrix solver (e.g. the MATLAB 'backslash' operator) has a computational complexity of $O(m^3N^3)$, where N is 1/2 the number of terms in the truncated series used to represent the prime function on the boundaries (the spectral truncation level). On the other hand, M2 has a computational complexity of $O((m+1)n\log n)$, where n is the number of collocation points on each boundary. For this check we set $N = 32$ in M1, which based on the result in the previous section is enough to achieve machine accuracy for the prime function, and we set $n = 128$ for M2. The result of the timing test is shown in Fig. 3.

10. Illustrative calculations

The introduction to this article surveyed the many varied uses of the S-K prime function in applications (see also Crowdy, 2008a, 2012). To give a flavour of the uses of the new computational tools we have described here, we present a sample of illustrative calculations.

10.1. *A calculus for ideal flows*

One of the authors (Crowdy, 2010a) has described how the S-K prime function can be used to build a complete 'calculus' for the study of two-dimensional ideal irrotational flows in general multiply connected domains. It turns out that the stream function for ideal irrotational flows past airfoils can be written explicitly in terms of the S-K prime function (Crowdy, 2010a) and, given this, the associated streamlines can be readily found by computing its contours. For example, the complex potential associated with uniform flow of speed U and angle of attack χ past any array of airfoils is (Crowdy, 2010a)

$$W(\zeta) = 2\pi U a \mathrm{i} \left[\mathrm{e}^{\mathrm{i}\chi} \frac{\partial G_0}{\partial \overline{\alpha}} - \mathrm{e}^{-\mathrm{i}\chi} \frac{\partial G_0}{\partial \alpha} \right]_{\alpha=\beta}, \tag{10.1}$$

where $z = z(\zeta)$ is the conformal map from a preimage circular domain D_ζ to the fluid region exterior to the airfoils having local behaviour

$$z = z(\zeta) \sim \frac{a}{\zeta - \beta} + \text{a locally analytic function} \tag{10.2}$$

as $\zeta \to \beta$, where β is the point mapping to infinity. Figure 4 shows the results for uniform flow past 7 circular airfoils, given by centres $\{s_j | j = 0, 1, \ldots, m\}$ and radii $\{r_j | j = 0, 1, \ldots, m\}$, where the circular domain D_ζ is defined by the simple Möbius mapping

$$\zeta = \frac{r_0}{z - s_0}. \tag{10.3}$$

Figure 4 also shows the flow corresponding to flow past 7 flat-plate airfoils where the conformal mapping from the circular domain just found is taken to be the radial slit map from Crowdy & Marshall (2006),

$$z = \frac{\omega(\zeta,\gamma)\omega(\zeta,1/\overline{\gamma})}{\omega(\zeta,\infty)\omega(\zeta,0)}, \tag{10.4}$$

where we chose $\gamma = -6.4$. The example shown in Fig. 4 was computed with the circle centres given by components of the vector

$$\begin{aligned}(0.605607 + 1.07664\mathrm{i}, &- 2.24299 - 0.672897\mathrm{i}, -2.33271 + 3.34206\mathrm{i}, 1.74953 - 1.41308\mathrm{i}, \\ &3.18505 + 2.71402\mathrm{i}, -4.66542 + 0.852336\mathrm{i}, 3.92523 - 0.964486\mathrm{i}),\end{aligned} \tag{10.5}$$

the respective circle radii given by the vector

$$(0.737804, 1.42496, 0.913313, 0.989207, 0.794918, 0.348927, 0.460223). \tag{10.6}$$

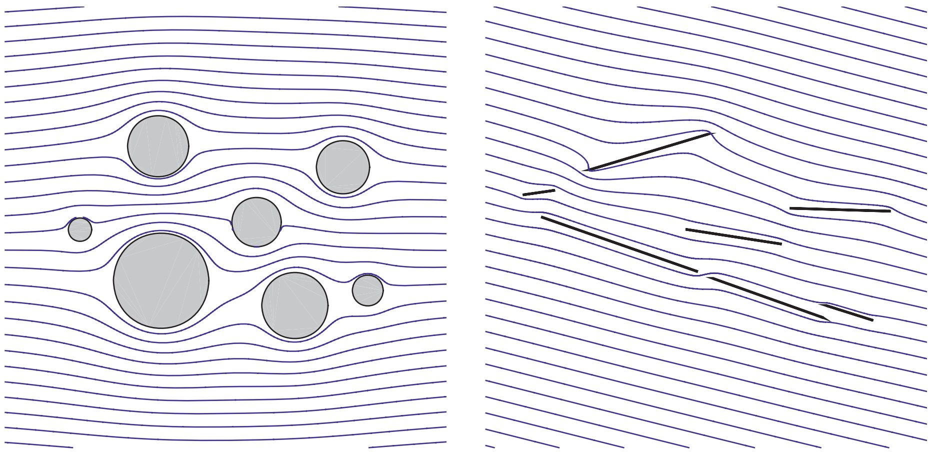

FIG. 4. Streamlines for uniform background flow past 7 circular islands (with zero round-island circulation) computed using the S-K prime function (shown left). The circular domain is then conformally mapped to a radial slit domain using (10.4) and the streamlines for uniform flow about the resulting configuration of flat plates is plotted (right).

10.2. *Kirchhoff–Routh path functions*

The trajectories of a point vortex around an island cluster can be written (Crowdy & Marshall, 2005) as the contours of the so-called Kirchhoff–Routh path function given explicitly in terms of the prime function as

$$H^{(z)}(z_\alpha, \overline{z_\alpha}) = -\frac{\Gamma^2}{8\pi} \log \left| \frac{\hat{\omega}(\alpha,\alpha)\overline{\hat{\omega}}(1/\alpha, 1/\alpha)}{\alpha^2 \omega(\alpha, 1/\overline{\alpha})\overline{\omega}(1/\alpha, \overline{\alpha}) z'(\alpha)^2} \right|, \tag{10.7}$$

where $z = z(\zeta)$ is the conformal map from a preimage circular domain D_ζ to the fluid region around the islands and $z_\alpha = z(\alpha)$ with $\hat{\omega}(\zeta, \alpha)$ defined through

$$\omega(\zeta, \alpha) = (\zeta - \alpha)\hat{\omega}(\zeta, \alpha). \tag{10.8}$$

The function $z'(\zeta)$ is the derivative of the conformal mapping. Armed with a code to compute the S-K prime function, the contours of (10.7) can be plotted easily. A set of vortex trajectories around seven islands inside the unit disc is given in Fig. 5. The interior circle centres are given by the vector

$$\begin{aligned}(-0.3501 + 0.4696\mathrm{i}, 0.12788 + 0.22222\mathrm{i}, -0.44654 - 0.2348\mathrm{i}, 0.014675 - 0.32704\mathrm{i},\\ 0.45493 - 0.48218\mathrm{i}, 0.64361 + 0.050314\mathrm{i}, 0.26205 + 0.62893\mathrm{i})\end{aligned} \tag{10.9}$$

and the respective circle radii by

$$(0.16902, 0.15199, 0.17288, 0.10482, 0.17794, 0.14256, 0.16282). \tag{10.10}$$

The conformal mapping is taken to be the identity mapping $z(\zeta) = \zeta$. This calculation can be accelerated by noticing that, for each evaluation point, the domain D_ζ is fixed meaning that recalculation of the functions $\{v_j(\zeta) | j = 1, ..., m\}$ is unnecessary.

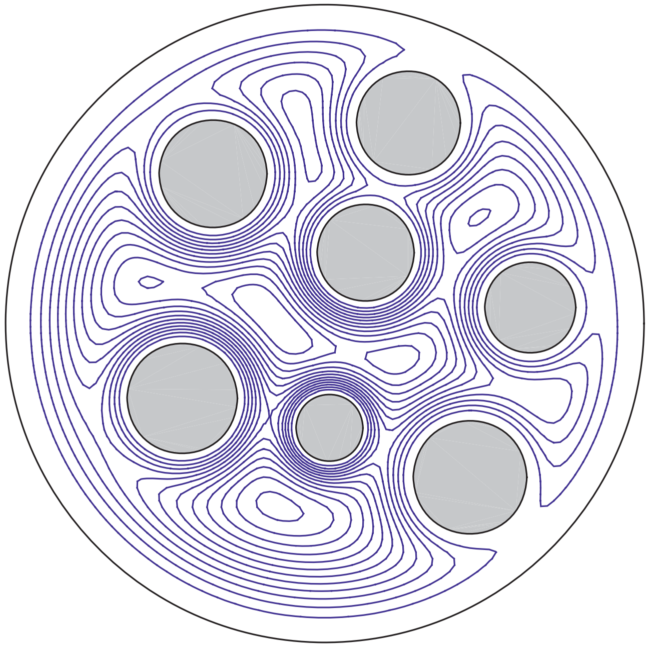

FIG. 5. Point vortex trajectories in a unit disc with seven impenetrable islands.

11. Discussion

The numerical scheme presented here is a natural development of the earlier work of Crowdy & Marshall (2007a): those authors were the first to present a constructive numerical technique for the S-K prime function associated to the Schottky doubles of multiply connected circular domains thereby avoiding the need to use poorly convergent infinite product representations (given by the classical authors) to evaluate it. Crowdy & Marshall (2007a) formulated a boundary value problem for $X(\zeta,\alpha)$ and used Fourier–Laurent representations of the requisite functions as the basis for computation of the prime function. Many of the steps of that original construction have been re-used here, but there is a crucial difference: here, as embodied in Theorems 6.1 and 6.2, we have shown, using the basic function theoretic objects of potential theory, that the prime function can be computed by solving a sequence of 'standard' modified Schwarz problems. This is a crucial realization since it turns the computation of the S-K prime function into a sequence of completely standard numerical steps for which convergence proofs of the methods are well established. Moreover, we have presented two novel and competitive numerical schemes for the solution of the relevant sequence of modified Schwarz problems. Readers should consult Crowdy (2016b) for future updates on the associated numerical software based on the methods expounded herein.

More generally, there has been a recent resurgence of interest in the advantages of characterizing special functions as the solutions of certain boundary value problems. Fokas & Glasser (2013) have recently shown that hypergeometric functions, and even the Riemann zeta function, can be defined as solutions of certain Neumann boundary value problems defined over domains bounded by the so-called Hankel contour. As a consequence, these authors were able to deduce some surprising new results about apparently well-known special functions. Similarly, Crowdy (2016a) has constructed so-called Ehrenpreis representations of trigonometric functions, the well-known gamma function and the

Weierstrass $\wp$-function borrowing ideas from a constructive transform approach to boundary value problems considered in Fokas & Kapaev (2003) and Crowdy (2015a,b); as a result, he was able to derive apparently new representations of the classical elliptic functions. In a similar vein, what we have demonstrated here is that the S-K prime function can also be characterized as the solution of a particular boundary value problem (of Dirichlet, or Schwarz, type) and that this viewpoint affords, as we have shown, significant advantages for its numerical evaluation. Indeed, following in exactly the spirit of the new transform method construction based on the analysis of global relations recently described in Crowdy (2016a), one of the constructions described herein (Method 1) is the very same transform approach adapted to the circular domain setting relevant herein.

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Appendix A. Proof of Theorem 6.1

Before giving the proof, we state and prove a useful Lemma.

LEMMA A1 Given $0 < \alpha < 1$ or $1 < \alpha < \infty$, it follows for all $|\zeta| = 1$ that

$$\arg \frac{\zeta\alpha}{(\zeta - \alpha)(\zeta - 1/\alpha)} = (2k + 1)\pi, \tag{A.1}$$

where k is any integer. In the case $\alpha = 1$ this is true provided $\zeta \neq 1$, and the expression has a removable singularity at $\zeta = 1$.

Proof. Let $R = \zeta\alpha/[(\zeta-\alpha)(\zeta - 1/\alpha)]$, and assume that $\alpha \neq 1$. A simple calculation shows that $R = \overline{R}$, so that R is real. Clearly R is never zero for any $|\zeta| = 1$, and we note that when $\zeta = 1$ we have $R < 0$. Thus $\arg R = (2k + 1)\pi$.

Now assume $\alpha = 1$. If $\zeta \neq 1$, then we still have $R = \overline{R}$. In fact R is again never zero for any $\zeta \neq 1$, and $R < 0$ when $\zeta = -1$. Picking a single branch of the logarithm, say $\arg[-1] = \pi$, we note the limit of $\arg R$ as ζ approaches 1 from either direction on the circle is π. The expression thus has a removable singularity at $\zeta = 1$. □

We can now give the proof of Theorem 6.1.

Proof of Theorem 6.1. Suppose that $0 < |\alpha| < \infty$, and define the function

$$Y_j(\zeta, \alpha) = X(\zeta, \alpha)\sqrt{-\phi_j'(\zeta)}\sqrt{-\phi_j'(\alpha)}. \tag{A.2}$$

Using the definition of the modified Green's function (5.6) above, we may write G_j as a ratio of the Y_j,

$$G_j(\zeta, \alpha) = \frac{1}{4\pi \mathrm{i}} \log\left[\frac{q_j^2}{|\alpha - \delta_j|^2} \frac{X(\zeta, \alpha)}{X(\zeta, \theta_j(1/\overline{\alpha}))}\right] = \frac{1}{4\pi \mathrm{i}} \log\left[\frac{Y_j(\zeta, \alpha)}{Y_j(\zeta, \theta_j(1/\overline{\alpha}))}\right]. \tag{A.3}$$

A calculation employing the conjugate relation (4.11) and the transformation property (4.7) shows that Y_j has the functional relation

$$\overline{Y_j}(\phi_j(\zeta), \phi_j(\alpha)) = Y_j(\zeta, \alpha), \tag{A.4}$$

which immediately implies that for all $\zeta \in C_j$

$$\mathrm{Im}\left[\log Y_j(\zeta, \alpha)\right] = 2\pi\, \mathrm{Re}\left[G_j(\zeta, \alpha)\right]. \tag{A.5}$$

Using the definition of Y_j and factoring $(\zeta-\alpha)^2$ from $X(\zeta,\alpha)$ we find

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[G_j(\zeta,\alpha)\right] + \arg\left[\frac{(\zeta-\delta_j)(\alpha-\delta_j)}{(\zeta-\alpha)^2}\right]. \tag{A.6}$$

In the case $j=0$, we factor G_0 via (5.11) for

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[\hat{G}_0(\zeta,\alpha)\right] + \arg\left[\frac{\zeta\alpha}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\right]. \tag{A.7}$$

Based on Lemma A1—and if necessary a rotation of C_0—the argument is a constant, and since $\hat{X}$ is determined up to an imaginary constant we may ignore it. On use of the previously derived relationship (5.13) from Lemma 5.1 between G_j, G_0 and v_j, we see for $\{\zeta\in C_j | j=1,\dots,m\}$ that

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[G_0(\zeta,\alpha)-v_j(\zeta)+v_j(\alpha)\right] + \arg\left[\frac{\alpha(\zeta-\delta_j)}{(\zeta-\alpha)^2}\right] \quad \text{when } \zeta\in C_j. \tag{A.8}$$

If the origin is in the fundamental domain, i.e. $q_j<|\delta_j|$ for all $j\in\{1,\dots,m\}$, factoring G_0 and v_j results in

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[\hat{G}_0(\zeta,\alpha)-\hat{v}_j(\zeta)+v_j(\alpha)\right] + \arg\left[\frac{\alpha(\zeta-\delta_j')}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\right]. \tag{A.9}$$

If instead $|\delta_j|\le q_j$, the origin is inside or on C_j, then v_j takes the alternate form given in (5.1), and the argument in the above expression is simply without the factor $(\zeta-\delta_j')$.

Now we would like to consider $\alpha=0$ or $\alpha=\infty$, given $0\in F$, or $q_j<|\delta_j|$ for all $j\in\{1,\dots,m\}$. Modifying Y_0 from above for our purposes,

$$Y_0(\zeta,\alpha) := X(\zeta,\alpha)/\zeta, \quad \text{when } \alpha\in\{0,\infty\}, \tag{A.10}$$

so that G_0 in terms of the prime function gives us

$$G_0(\zeta,\alpha) = \frac{1}{4\pi\mathrm{i}}\log\left[\frac{X(\zeta,\alpha)}{X(\zeta,1/\overline{\alpha})}\right] = \frac{1}{4\pi\mathrm{i}}\log\left[\frac{Y_0(\zeta,\alpha)}{Y_0(\zeta,1/\overline{\alpha})}\right]. \tag{A.11}$$

Using the now familiar transformation property and the conjugate relation, it is easy to show Y_0 has the functional relation $\overline{Y_0}(1/\zeta,1/\alpha)=Y_0(\zeta,\alpha)$. This relation immediately implies that for $\zeta\in C_0$,

$$4\pi\mathrm{i}G_0(\zeta,\alpha) = \log Y_0(\zeta,\alpha) - \log\overline{Y_0(\zeta,\alpha)}, \tag{A.12}$$

and, given $\operatorname{Im}[G_0]=0$ when $\zeta\in C_0$, we have hence

$$\operatorname{Im}\left[\log Y_0(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[G_0(\zeta,\alpha)\right], \tag{A.13}$$

where after factoring both sides, applying Lemma A1 and ignoring the constant we find

$$\operatorname{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = 2\pi \operatorname{Re}\left[\hat{G}_0(\zeta,\alpha)\right]. \tag{A.14}$$

Furthermore, the transformation property (4.7) implies

$$Y_0(\theta_j(\zeta),\alpha) = \frac{\theta_j(\zeta)}{\zeta} H_j(\zeta,\alpha) Y_0(\zeta,\alpha), \tag{A.15}$$

and it follows that, for $\zeta \in \{C_j : j = 1,\ldots,m\}$,

$$\mathrm{Im}\left[\log Y_0(\zeta,\alpha)\right] = 2\pi G_0(\zeta,\alpha) + \frac{1}{2\mathrm{i}} \log H_j(1/\overline{\zeta}, 1/\overline{\alpha}) - \frac{1}{2\mathrm{i}} \log |\zeta|^2. \tag{A.16}$$

It is not apparent that the right-hand side of this equation is a real quantity. To see that it is we begin by recalling

$$G_0(\zeta,\alpha) = \mathrm{Re}\,[G_0(\zeta,\alpha)] + \mathrm{i}\Gamma_{0j} \qquad \text{for} \qquad \zeta \in C_j,\ j > 0. \tag{A.17}$$

From Crowdy & Marshall (2007b) we have that

$$\Gamma_{0j} = \frac{1}{4\pi} \log|\exp\left(2\pi\mathrm{i}\left[v_j(1/\overline{\alpha}) - v_j(\alpha)\right]\right)| = \mathrm{Im}\left[v_j(\alpha)\right], \tag{A.18}$$

where the second equality follows from the property $v_j(\zeta) = \overline{v_j}(1/\zeta)$. Using these facts along with the imaginary constant $\tau_{jj} = v_j(\theta_j(\zeta)) - v_j(\zeta)$, we can show for ζ on C_j that (A.16) reduces to

$$\mathrm{Im}\left[\log Y_0(\zeta,\alpha)\right] = 2\pi\ \mathrm{Re}\left[G_0(\zeta,\alpha) + v_j(\alpha) - v_j(\zeta)\right] + \frac{1}{2\mathrm{i}} \log\left[\frac{q_j^2\overline{\zeta}^2}{(\overline{\zeta} - \overline{\delta_j})} \frac{1}{|\zeta|^2}\right], \tag{A.19}$$

which now clearly takes on real values, since a simple further calculation shows

$$\frac{1}{2\mathrm{i}} \log\left[\frac{q_j^2\overline{\zeta}^2}{(\overline{\zeta} - \overline{\delta_j})} \frac{1}{|\zeta|^2}\right] = \arg\left[\frac{\zeta - \delta_j}{\zeta}\right]. \tag{A.20}$$

We thus rewrite (A.16) for $\zeta \in \{C_j : j = 1,\ldots,m\}$, in the form

$$\mathrm{Im}\left[\log Y_0(\zeta,\alpha)\right] = 2\pi\ \mathrm{Re}\left[G_0(\zeta,\alpha) + v_j(\alpha) - v_j(\zeta)\right] + \arg\left[\frac{\zeta - \delta_j}{\zeta}\right]. \tag{A.21}$$

The conclusion follows after applying the definition of Y_0 and factoring. □

Appendix B. Proof of Theorem 6.2

Proof. We begin with the function definition

$$Y_{jk}(\zeta,\alpha) = X(\zeta,\alpha)\sqrt{-\phi_j'(\zeta)}\sqrt{-\phi_k'(\alpha)} \tag{B.1}$$

where we immediately see that

$$G_k(\zeta,\alpha) = \frac{1}{4\pi\mathrm{i}} \log\left[\frac{Y_{jk}(\zeta,\alpha)}{Y_{jk}(\zeta,\overline{\phi_k(\alpha)})}\right]. \tag{B.2}$$

Consider the case α and ζ are on the same boundary, C_k. Since Y_{kk} is the same as function Y_k given in the proof of Theorem 6.1, we have the function identity

$$\overline{Y_{kk}}(\phi_k(\zeta),\phi_k(\alpha)) = Y_{kk}(\zeta,\alpha). \tag{B.3}$$

We then recall G_k is identically zero for all $\zeta \in C_k$ to find

$$\mathrm{Im}\left[\log Y_{kk}(\zeta,\alpha)\right] = 0, \tag{B.4}$$

which means that, for $\zeta,\alpha \in C_k$,

$$\mathrm{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = \arg\left[\frac{(\zeta-\delta_k)(\alpha-\delta_k)}{(\zeta-\alpha)^2}\right]. \tag{B.5}$$

By a scaling and rotation of C_k, Lemma A1 applies, and since $\hat{X}$ is determined up to an imaginary constant, we may in this case take $\mathrm{Im}[\hat{X}]$ to be zero.

To consider ζ and α on different boundaries, we note that applying the transformation identity (4.7) it is possible to show that

$$\overline{Y_{jk}}(\phi_j(\zeta),\phi_k(\alpha)) = H_{jk}(\zeta,\alpha)Y_{jk}(\zeta,\alpha) \tag{B.6}$$

for $j \neq k$, where

$$H_{jk}(\zeta,\alpha) = \begin{cases} \exp\left(4\pi\mathrm{i}\left[-\overline{v_k}(1/\zeta)+\overline{v_k}(1/\alpha)+\frac{1}{2}\overline{\tau_{kk}}\right]\right) & j=0,\ k>0 \\ \exp\left(4\pi\mathrm{i}\left[\overline{v_j}(1/\zeta)-\overline{v_k}(1/\zeta)-\overline{v_j}(\overline{\alpha})\right.\right. & \\ \qquad\left.\left.+\overline{v_k}(1/\alpha)+\frac{1}{2}\left(\overline{\tau_{jj}}+\overline{\tau_{kk}}\right)\right]\right) & k,j>0,\ k\neq j, \\ \exp\left(4\pi\mathrm{i}\left[\overline{v_j}(1/\zeta)-\overline{v_j}(\overline{\alpha})+\frac{1}{2}\overline{\tau_{jj}}\right]\right) & j>0,\ k=0. \end{cases} \tag{B.7}$$

Applying the functional relation (B.6) we find

$$\mathrm{Im}\left[\log Y_{jk}(\zeta,\alpha)\right] = \frac{1}{2\mathrm{i}}\log\overline{H_{jk}(\zeta,\alpha)}, \tag{B.8}$$

where the definition of Y_{jk} and factoring X gives us

$$\mathrm{Im}\left[\log \hat{X}(\zeta,\alpha)\right] = \frac{1}{2\mathrm{i}}\log\overline{H_{jk}(\zeta,\alpha)} + \arg\left[\frac{(\zeta-\delta_j)(\alpha-\delta_k)}{(\zeta-\alpha)^2}\right]. \tag{B.9}$$

It is possible to appeal to Lemma A1 to drop the constant, but we will not do that here, since the resulting expression needs to be single valued, and we will use the factors in the constant argument expression to achieve proper branch cuts. So using properties (5.4) and (5.5) of the first kind integrals shows that for $k, j > 0$ and $k \neq j$ we may write

$$\frac{1}{2\mathrm{i}} \log \overline{H_{jk}(\zeta,\alpha)} = 2\pi \,\mathrm{Re}\left[\hat{v}_k(\zeta) + v_j(\alpha) - \hat{v}_j(\zeta) - v_k(\alpha)\right] + \arg\left[\frac{\zeta-\delta_k}{\zeta-\delta_k'}\frac{\zeta-\delta_j'}{\zeta-\delta_j}\right], \tag{B.10}$$

where the case when $j = 0$ follows in the obvious way, as does the case when $k = 0$. Substitution and consideration of the various domain configurations completes the proof. □

Appendix C. Numerical continuation

This appendix gives details of the numerical continuation procedure used once the method of Section 7 or the method of Section 8 has been used to find the boundary values of $\hat{X}$. Given this data, we need a way to evaluate $\hat{X}$ for ζ inside F.

One might be naturally inclined to apply the Cauchy integral formula, but it is well known this is numerically unstable, especially for evaluation points near the boundary. Alternatively we apply a type of barycentric interpolation derived from applying Cauchy's theorem. This method was first discussed in Ioakimidis *et al.* (1991) and was shown to be numerically stable in general in Higham (2004). The method has been expounded elsewhere, so we will not give details here; see, e.g. Trefethen & Weideman (2014) or Nasser (2015) for more information.

We have created the numerical continuation of $\hat{X}(\zeta,\alpha)$ and consequently $X(\zeta,\alpha)$, into D_ζ, but we have not addressed how to evaluate these functions in the closure of D_ζ'. To do this, we use a functional relation of X, which comes directly from the relation (4.11),

$$X(\zeta,\alpha) = (\zeta\alpha)^2\overline{X(1/\overline{\zeta}, 1/\overline{\alpha})}. \tag{C.1}$$

Using the relationship between X and $\hat{X}$ from above this reduces to

$$\hat{X}(\zeta,\alpha) = \overline{\hat{X}(1/\overline{\zeta}, 1/\overline{\alpha})}. \tag{C.2}$$

This tells us exactly the form of the solution we need to evaluate points outside D_ζ. In what follows we will use with impunity the fact, following from (4.10), that

$$X(\zeta,\alpha) = X(\alpha,\zeta). \tag{C.3}$$

The discussion considers two cases, as in Section 6: either α is in the fundamental domain or α is on a boundary of F.

C.1. *For α in the fundamental domain*

Suppose $\alpha \in \{F \setminus C_0\}$. We then solve the Schwarz problem for $\hat{X}(\zeta,\alpha)$ as described in Section 7.3 and apply continuation by Cauchy's theorem as above, which allows us to evaluate $\hat{X}$ for all ζ in the closure of D_ζ. To evaluate the function at values of ζ in the closure of D_ζ', we would like to use $\hat{X}(\zeta, 1/\overline{\alpha})$. So we solve again the same boundary value problem but with α replaced by $1/\overline{\alpha}$. Thus for ζ in the closure of F, we have

$$\hat{X}(\zeta,\alpha) = \begin{cases} \hat{X}(\zeta,\alpha) & |\zeta| \le 1, \\ \overline{\hat{X}(1/\overline{\zeta},1/\overline{\alpha})} & |\zeta| > 1. \end{cases} \tag{C.4}$$

Hopefully the abuse of notation here and below does not cause confusion; the $\hat{X}$ on the left of the equality refers to the desired solution and the two uses of $\hat{X}$ on the right refer to computed solutions.

When α is on the unit circle, it is obviously still in the fundamental domain, but especially convenient is that $\alpha = 1/\overline{\alpha}$. Thus we need to solve only one boundary value problem for $\hat{X}$, i.e.

$$\hat{X}(\zeta,\alpha) = \begin{cases} \hat{X}(\zeta,\alpha), & |\zeta| \le 1, \\ \overline{\hat{X}(1/\overline{\zeta},\alpha)}, & |\zeta| > 1. \end{cases} \tag{C.5}$$

C.2. *For α on a boundary*

Now we assume α is on the boundary of the fundamental domain. There are two very similar but distinct cases for numerical continuation: α on boundary circles in D_ζ and α on boundary circles in D'_ζ.

C.2.1. *For α on an inner boundary.* Suppose that $\alpha \in C_j$ for some $j \in \{1,\dots,m\}$. Then we know how to compute $\hat{X}(\zeta,\alpha)$ for values of ζ less than unit modulus in F, and we have that $\theta_j(1/\overline{\alpha}) = \alpha$. Using the transformation property (4.7),

$$X(\theta_j(1/\overline{\alpha}),\zeta) = H_j(1/\overline{\alpha},\zeta)X(1/\overline{\alpha},\zeta), \tag{C.6}$$

so that

$$\hat{X}(\zeta,1/\overline{\alpha}) = \left(\frac{\zeta-\alpha}{\zeta-1/\overline{\alpha}}\right)^2 \frac{\hat{X}(\zeta,\alpha)}{H_j(1/\overline{\alpha},\zeta)} \tag{C.7}$$

is then used for $|\zeta| > 1$ in (C.4). Note that H_j here involves the function v_j, which has already been computed to find $\hat{X}$.

C.2.2. *For α on an outer boundary.* Suppose that $\alpha \in C'_j$ for some $j \in \{1,\dots,m\}$. Then we know how to compute $\hat{X}(1/\overline{\zeta},1/\overline{\alpha})$ for values of ζ greater than unit modulus in F, and we have that $\theta_j(\alpha) = 1/\overline{\alpha}$. Applying again the transformation property (4.7) we see

$$X(\theta_j(\alpha),\zeta) = H_j(\alpha,\zeta)X(\alpha,\zeta) \tag{C.8}$$

so that

$$\hat{X}(\zeta,\alpha) = \left(\frac{\zeta-1/\overline{\alpha}}{\zeta-\alpha}\right)^2 \frac{\hat{X}(\zeta,1/\overline{\alpha})}{H_j(\alpha,\zeta)} \tag{C.9}$$

is then used for $|\zeta| \le 1$ in (C.4). Not surprisingly, this is exactly the same formula for the case $\alpha \in C_j$ in terms of the location of α. It is a reminder that the transformation property relates points on the inner and outer circles of the fundamental domain.

REMARK C1 The case when $\alpha = \infty$ and $\alpha \in C_j'$, i.e. when $|\delta_j| = q_j$ (or $0 \in C_j$), is still not defined, since $H_j(\infty, \zeta)$ is identically zero. The failure of the transformation property (4.7) in this case is a direct result of our choice of normalization which places the pole of the prime function at infinity. The situation precludes computing $\omega(\zeta, 0)$ for $|\zeta| > 1$, but a simple conformal automorphism of the unit disc which takes, say, $\delta_j \mapsto 0$ for some choice of $0 < j \le m$ suffices to get around this problem.

Appendix D. Correction for the parameter near a boundary

In the statement of the boundary problem in Theorem 6.1, it should be clear that if the parameter α is 'near enough' to an inner boundary, say C_k, then the term $(\zeta - \alpha)$ will produce 'near singular' behaviour in the function A_j from Theorem 6.1 when $j = k$. (The measurement 'near enough' will be made a bit more precise below.) We take 'near singular' behaviour to be such that the curvature of the function in question becomes too great to be handled neatly by the trapezoidal rule using a given number of collocation points. Moreover, the numerical solution to $\hat{G}_0$ will suffer for the same reason. If instead $1/\overline{\alpha}$ is 'near enough' an inner boundary, then the term $(\zeta - 1/\overline{\alpha})$ will cause the same problems. We fix this with an additional factorization of the prime function.

The part of the prime function without the zero factor $(\zeta - \alpha)$, call it $\hat{\omega}$, has the infinite product representation (Baker, 1897)

$$\hat{\omega}(\zeta, \alpha) = \prod_{\theta \in \Theta''} \frac{(\zeta - \theta(\alpha))(\alpha - \theta(\zeta))}{(\zeta - \theta(\zeta))(\alpha - \theta(\alpha))}, \tag{D.1}$$

where Θ'' is the Schottky group mentioned in Section 4 minus inverses and the identity. Now suppose α (or $1/\overline{\alpha}$) is 'near enough' C_k to cause the problems mentioned in the previous paragraph. We pull two factors involving θ_k out of this product representation to write

$$\omega(\zeta, \alpha) = (\zeta - \alpha)\hat{\omega}(\zeta, \alpha) = (\zeta - \alpha)\frac{(\zeta - \theta_k(\alpha))}{(\zeta - \theta_k(\zeta))}\tilde{\omega}(\zeta, \alpha), \tag{D.2}$$

from which we find an altered version of the modified Green's function for C_0,

$$G_0(\zeta, \alpha) = \tilde{G}_0(\zeta, \alpha) + \frac{1}{2\pi \mathrm{i}} \log\left[\frac{(\zeta - \alpha)(\zeta - \theta_k(\alpha))}{(\zeta - 1/\overline{\alpha})(\zeta - \theta_k(1/\overline{\alpha}))}\right], \tag{D.3}$$

and the modified version of the square of the prime function

$$X(\zeta, \alpha) = (\zeta - \alpha)^2 \frac{(\zeta - \theta_k(\alpha))^2}{(\zeta - \theta_k(\zeta))^2}\tilde{X}(\zeta, \alpha). \tag{D.4}$$

The functions $\tilde{G}_0$ and $\tilde{X}$ are analytic in the fundamental domain. For the numerical solution of G_0, it should be immediately clear that the term $(\zeta - \alpha)$ is balanced while traversing C_k by the term $(\zeta - \theta_j(1/\overline{\alpha}))$, in the sense of singularity subtraction, as is the term $(\zeta - 1/\overline{\alpha})$ balanced by $(\zeta - \theta_j(\alpha))$. (This can be seen by considering inversion in the circle C_k of the point α or $1/\overline{\alpha}$.) It follows that we will now be solving for $\tilde{X}$, and the equations of Theorem 6.1 must be restated in the form

$$\operatorname{Im}\left[\log \tilde{X}(\zeta,\alpha)\right] = \begin{cases} 2\pi \operatorname{Re}\left[\tilde{G}_0(\zeta,\alpha)\right] + A_0(\zeta,\alpha) & \zeta \in C_0 \\ 2\pi \operatorname{Re}\left[\tilde{G}_0(\zeta,\alpha) - \hat{v}_j(\zeta) + v_j(\alpha)\right] + A_j(\zeta,\alpha) & \zeta \in C_j,\ j > 0; \end{cases} \tag{D.5}$$

where

$$A_0(\zeta,\alpha) = \begin{cases} \arg\left[\dfrac{(\zeta-\theta_k(\zeta))^2}{(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}\right] & \alpha \in \{0,\infty\}, \\ \arg\left[\dfrac{\alpha\zeta(\zeta-\theta_k(\zeta))^2}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}\right] & 0 < |\alpha| < \infty, \end{cases} \tag{D.6}$$

and

$$A_j(\zeta,\alpha) = \begin{cases} \arg\left[\dfrac{(\zeta-\delta_j')(\zeta-\theta_k(\zeta))^2}{\zeta(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}\right] & q_j < |\delta_j|,\ \alpha \in \{0,\infty\}, \\ \arg\left[\dfrac{\alpha(\zeta-\delta_j')(\zeta-\theta_k(\zeta))^2}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}\right] & q_j < |\delta_j|,\ 0 < |\alpha| < \infty, \\ \arg\left[\dfrac{\alpha(\zeta-\theta_k(\zeta))^2}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}\right] & 0 \le |\delta_j| \le q_j. \end{cases} \tag{D.7}$$

The key observation for the numerical solution of $\tilde{X}$ is that now the term $(\zeta-\theta_k(1/\overline{\alpha}))$ is in the proper place to correct the singular behaviour, on the traversal of C_k, of $(\zeta-\alpha)$ in the argument function. Similarly for the pairing for the $(\zeta-1/\overline{\alpha})$ factor. The two additional singularities inside C_k do not introduce a branch cut across the circle C_k, since the (double strength) fixed point of the term $(\zeta-\theta_k(\zeta))^2$ inside C_k pairs with both of the new singularities. The upshot being there is still a choice of branch cuts for A_0 and A_j which allow the functions to remain single-valued.

In the case α is too near more than one inner circle, the fix is similar, if only slightly more complicated in the number of terms to track. Let E be the set of indices of such circles. We then define the function

$$\psi(\zeta,\alpha) = \prod_{k\in E} \frac{\zeta-\theta_k(\alpha)}{\zeta-\theta_k(\zeta)}, \tag{D.8}$$

from which we now write

$$X(\zeta,\alpha) = (\zeta-\alpha)^2\psi(\zeta,\alpha)^2\tilde{X}(\zeta,\alpha). \tag{D.9}$$

The modified Green's function for C_0 now takes the form

$$G_0(\zeta,\alpha) = \tilde{G}_0(\zeta,\alpha) + \frac{1}{2\pi\mathrm{i}}\log\left[\frac{(\zeta-\alpha)\psi(\zeta,\alpha)}{(\zeta-1/\overline{\alpha})\psi(\zeta,1/\overline{\alpha})}\right], \tag{D.10}$$

and so following the definition of the function

$$\xi(\zeta,\alpha)=\prod_{k\in E}\frac{(\zeta-\theta_k(\zeta))^2}{(\zeta-\theta_k(\alpha))(\zeta-\theta_k(1/\overline{\alpha}))}, \tag{D.11}$$

we jump directly to the modified form of the equations for Theorem 6.1:

$$\operatorname{Im}\left[\log\tilde{X}(\zeta,\alpha)\right]=\begin{cases}2\pi\operatorname{Re}\left[\tilde{G}_0(\zeta,\alpha)\right]+A_0(\zeta,\alpha) & \zeta\in C_0,\\ 2\pi\operatorname{Re}\left[\tilde{G}_0(\zeta,\alpha)-\hat{v}_j(\zeta)+v_j(\alpha)\right]+A_j(\zeta,\alpha) & \zeta\in C_j,\ j>0;\end{cases} \tag{D.12}$$

where

$$A_0(\zeta,\alpha)=\begin{cases}\arg\left[\xi(\zeta,\alpha)\right] & \alpha\in\{0,\infty\},\\ \arg\left[\dfrac{\alpha\zeta}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\,\xi(\zeta,\alpha)\right] & 0<|\alpha|<\infty,\end{cases} \tag{D.13}$$

and

$$A_j(\zeta,\alpha)=\begin{cases}\arg\left[\frac{(\zeta-\delta_j')}{\zeta}\,\xi(\zeta,\alpha)\right] & q_j<|\delta_j|,\ \alpha\in\{0,\infty\},\\ \arg\left[\frac{\alpha(\zeta-\delta_j')}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\,\xi(\zeta,\alpha)\right] & q_j<|\delta_j|,\ 0<|\alpha|<\infty,\\ \arg\left[\dfrac{\alpha}{(\zeta-\alpha)(\zeta-1/\overline{\alpha})}\,\xi(\zeta,\alpha)\right] & 0\le|\delta_j|\le q_j.\end{cases} \tag{D.14}$$

Numerical experiments have shown that ‘near enough’ in the bounded domain D_ζ is in general a distance between a circle and the parameter of approximately 0.1—in the vast majority of cases a marked decrease in the numerical accuracy can be seen at this distance. Since a plot of A_k starts to visibly show a change in curvature when α has a distance of about 0.15 from C_k, we have settled on this value in the software Kropf (2015) to trigger the use of the above correction.